Advanced Logic — Homework assignement

Tableaux and Natural Deduction for First-Order Logic

Total: 20 points

Instructions. You may consult your notes and the course material. Justify every step. In natural deduction proofs, cite the rule and the line numbers used. In tableaux, number every line and indicate which rule and line produced it. Discussion questions require careful, precise writing; they are graded on the quality of your reasoning, not merely on arriving at the right answer. Suggested length for each discussion question: 150–250 words.

Part A — Tableaux for First-Order Logic *(6 points)*

Exercises *(3 points)*

Exercise 1 *(1 point)*

Build a tableau to show that the following sequent (i.e. an expression $\Gamma \vdash \varphi$, where Γ is a set of premises and φ is a conclusion) is valid:

$$\forall x(Fx \to Gx),\ \exists x\, Fx \ \vdash\ \exists x\, Gx$$

Exercise 2 *(1 point)*

Build a tableau to show that the following sequent is valid:

$$\neg\exists x(Fx \wedge Gx) \ \vdash\ \forall x(Fx \to \neg Gx)$$

Exercise 3 *(1 point)*

Build a tableau to show that the following sequent is *invalid.* Read off a countermodel from the open branch(es), and verify that it is indeed a countermodel.

$$\forall x(Fx \to Gx),\ \exists x\, Gx \ \vdash\ \exists x\, Fx$$

Discussion *(3 points)*

After completing Exercise 3, you obtained a countermodel by reading off information from an open branch. But why should we trust this procedure? Answer the following questions.

(a) Explain what it means for the tableau method to be *sound.* State the property precisely in your own words. *(1 point)*

(b) Now use soundness to explain why the structure you read off the open branch of a *completed* tableau is guaranteed to be a genuine countermodel—not just a convenient guess. Your explanation should articulate the connection between the following two facts: (i) no tableau rule can close the branch, and (ii) every formula on the branch is satisfied by a single valuation. *(2 points)*

Part B — Natural Deduction for First-Order Logic *(6 points)*

Exercises *(3 points)*

Exercise 4 *(1 point)*

Prove the following sequent in natural deduction:

$$\exists x\, Fx \ \vdash\ \neg\forall x\, \neg Fx$$

Exercise 5 *(1 point)*

Prove the following sequent in natural deduction:

$$\forall x(Fx \to Gx),\ \forall x(Gx \to Hx)\ \vdash\ \forall x(Fx \to Hx)$$

Exercise 6 *(1 point)*

Prove the following sequent in natural deduction:

$$\forall x(Fx \to Gx)\ \vdash\ \neg\exists x(Fx \wedge \neg Gx)$$

Discussion *(3 points)*

This question concerns the *eigenvariable condition*—the restriction that governs the rules ∀I and ∃E.

(a) Consider your proof of Exercise 5. Identify every step at which a variable restriction (the eigenvariable condition) applies. For each such step, state the restriction precisely and verify that your proof satisfies it. If a quantifier rule in your proof carries *no* eigenvariable restriction, say so and explain why. *(1 point)*

(b) Now suppose we *drop* the eigenvariable condition entirely: any constant may be used anywhere, with no freshness or independence requirement. Construct a concrete example of an *invalid* sequent that becomes "provable" under the unrestricted rule. Your example should include:

- the invalid sequent,
- the spurious "proof" that uses the unrestricted rule,
- a countermodel showing that the sequent is indeed invalid.

(1 point)

(c) In one or two sentences, explain the logical purpose of the eigenvariable condition: what does it guarantee about the constant over which we generalise (in ∀I) or which we use as a witness (in ∃E)? *(1 point)*

Part C — The Ontology of Holes *(8 points)*

In this problem, we take a look at some slightly marginal (due to their emptiness) but nonetheless amusing entities: holes. It's going to be superficial, but that's the point.

C.1 "Being a hole" is relational *(1 point)*

Even though "is a hole in" is fundamentally relational, a monadic predicate Hx can be defined from it. Give such a definition. It should have a biconditional form:

$$\forall x \big(Hx \; \leftrightarrow \; \underbrace{\qquad\qquad\qquad\qquad}_{\text{fill in}} \big)$$

C.2 The fundamental axiom of holes ontology *(1 point)*

The entity on which a hole depends for its existence is not itself a hole. Indeed, a hole cannot be holed. Using Hxy and Hx, formulate the fundamental axiom of hole theory, which states that the host of a hole is not a hole. Your formula should begin with two universal quantifiers.

C.3 A theorem of holes ontology *(3 points)*

Express the following fact in first-order logic, and derive it in natural deduction. You should use only the definition from C.1 and the axiom from C.2 as premises.

Hxy describes an irreflexive relation; that is, nothing is a hole in itself:

$$\forall x \, \neg Hxx$$

C.4 Discussion *(3 points)*

(a) Consider the fundamental axiom you formulated in C.2. What is its status? For instance, one might hold that it is a *logical truth* (true in every model), an *empirical generalisation* (true as a matter of fact, but conceivably false), a *stipulative definition* (true by decision about how we use the word "hole"), or something else entirely. Could someone coherently deny it? Justify your answer, and if possible give an example that supports your position. *(1 point)*

(b) Transitivity. Formalise the following claim in first-order logic:

Hxy is transitive: if x is a hole in y and y is a hole in z, then x is a hole in z.

Does this follow from the definition (C.1) and the fundamental axiom (C.2) alone? Attempt a natural deduction proof. Whether or not you succeed, discuss carefully what your proof attempt reveals about the interaction between the definition and the axiom. In particular: is the conjunction $Hxy \wedge Hyz$ consistent with the axiom? What does your answer tell us about the ontological structure that the axiom imposes on holes? *(2 points)*