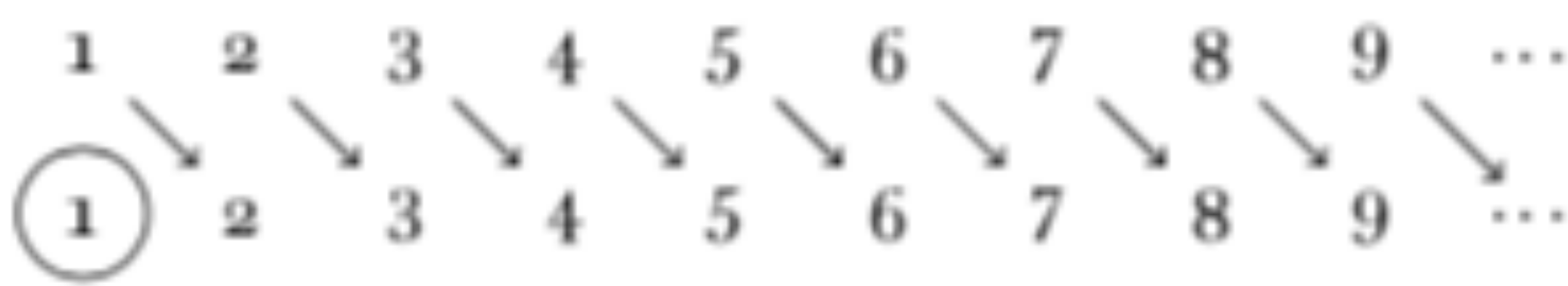
Logic, sets, and the infinite

I Infinite sets

Hilbert's Grand Hotel

- First, let's imagine a hotel with a finite number of rooms, each of which is occupied by a tourist.
- If we ask tourists to change rooms, it will always be the case that when a tourist moves to a new room, say n+1, the tourist in that room will have to find a new empty room to occupy... and so will move to room n+2, for example.
- But then the tourist in the last room, say z, will have to move to the first, in room 1.
- So tourists have to swap rooms — that's called a permutation in mathematics.
- **This method cannot be used to find a new room for a newcomer, since by definition the newcomer does not yet have a room, and so cannot be permuted with the occupant of another room.**

- On the other hand, let's imagine that the number of rooms is infinite.
- In this case, there's a way to make room for a newcomer: ask each tourist to move one room to the right:
- The function f : $n \longrightarrow n+1$ assigns a room to each tourist, clearing room 1 for the newcomer as follows :

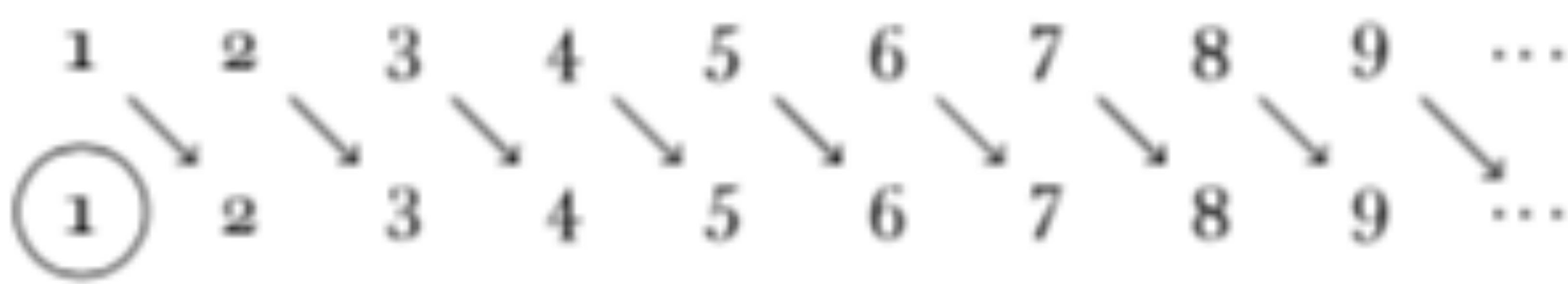


Equinumerosity

- Two sets A and B are **equinumerous** iff there exists a one-to-one correspondence (or bijection) between them:
    - that is, if there exists a function from A to B such that for every element y of B, there is exactly one element x of A with $f(x) = y$.
- Equinumerous sets are said to have the same **cardinality** (number of elements); but note that we do not need to use the concept « number of elements » in order to define « same cardinality »: we just need the concept of a bijection.
- The binary relation « being equinumerous » is an **equivalence relation**, which means that it is reflexive, transitive and symmetrical.

Dedekind and the infinite

- We can use the concept of equinumerosity in order to give a precise meaning to Hilbert's Grand Hotel intuition:
- A set A is Dedekind-infinite iff some proper subset B of A is equinumerous to A.
  - This means that there exists a bijection from A onto some proper subset B of A ...
  - ... which is exactly the situation described by Hilbert:



- Note that this definition does not use the concept of a natural number: it is a purely set-theoretic definition.

II Sets and numbers

The Frege / Russell approach

- Frege, *The Foundations of Arithmetic*:
  - Hume's principle: if there are as many Fs as Gs, the number of Fs = the number of Gs.
  - In this case, there is a bijection between the set of Fs and the set of Gs, and the two sets are said to have the same cardinal or to be equinumerous.
- How, then, to define the successor of a number?
  - Russell: The number of Gs is the successor of the number of Fs if (i) there is an object o that is G and (ii) if we set aside this object o, the number of Gs that are not o = the number of Fs.

Number as equivalence classes

- We already saw that the equinumerosity binary relation was an equivalence relation.
- For this reason, it is possible to define numbers as equivalence classes of equinumerous sets; that's what Frege and Russell propose :
  - 0 is the set of all sets equinumerous to $\varnothing$, which can itself be defined as $\{x \mid x \neq x\}$;
  - Then we can create a new set having just one element, let us say, $\{\varnothing\}$, and define 1 as the set of all sets equinumerous to $\{\varnothing\}$;
  - And we can define 2 as the set all all sets equinumerous to $\{\varnothing, \{\varnothing\}\}$
  - …etc

Are there enough such numbers?

- On this view, **n** is defined, as the set of sets having the same cardinal as a certain set having *n elements*.
- Since **n** can be arbitrarily large, how can we prove the existence of a set with *n* elements?
- Frege's answer in *Foundations of Arithmetic*:
  - The existence of 0 can be shown
  - If we consider the set with 0 as its only element, we have a set with 0's successor (i.e. 1) as its cardinal
  - In general: suppose we've already shown the existence of numbers between 0 and n-1; these numbers constitute a set with n elements; we've therefore proved the existence of a set with n elements, and therefore of **n**.

Von Neumann's construction

- Von Neumann's construction takes up Frege's idea:
  - 0 is defined as $\varnothing$
  - The successor function S is recursively defined thus:
    - For all n, $Sn = n \cup \{n\}$
  - So we have:

$$\begin{aligned}
0 &= \{\} &&= \varnothing, \\
1 &= \{0\} &&= \{\varnothing\}, \\
2 &= \{0,1\} &&= \{\varnothing,\{\varnothing\}\}, \\
3 &= \{0,1,2\} &&= \{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}.
\end{aligned}$$

Is there an infinite set?

- Now let us consider the set such that:
  - $\varnothing \in A$
  - For all x, if $x \in A$, then $x \cup \{x\} \in A$
- The function : $x \longrightarrow x \cup \{x\}$ defined on A is injective, which shows that if A exists, it is infinite.
- This set satisfies all Peano axioms.
- But there is no obvious proof that it does exist, hence the **Axiom of infinity, which states that there is indeed such a set A.**

III Infinite cardinals

Cardinal numbers

- The concept of cardinality can be introduced by an axiom in modern set theory:
    - The sets A and B are equinumerous if and only if Card(A) = Card(B).
- Some cardinals are countable and finite; some are countable and infinite; and some are not countable and infinite.

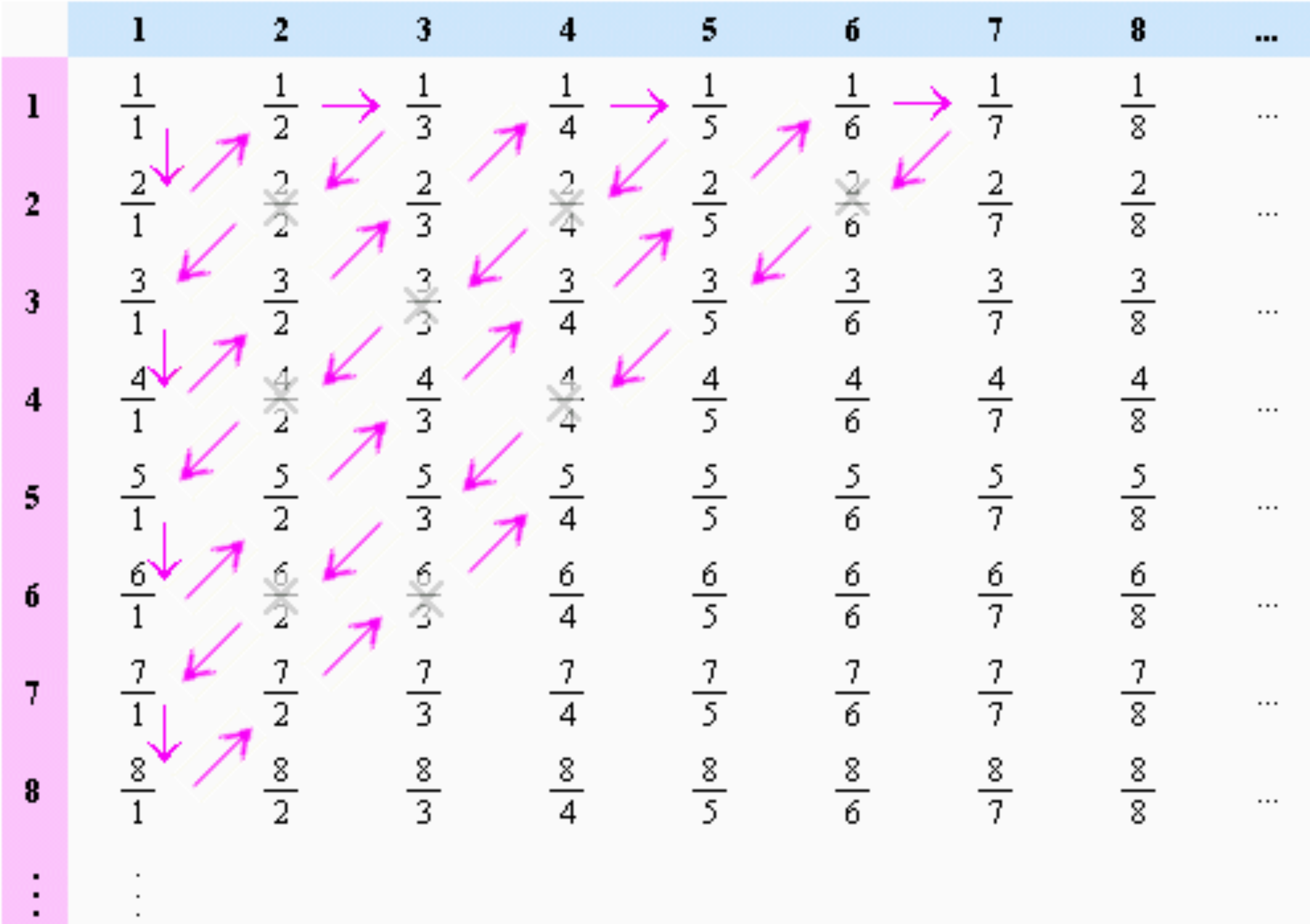
Countable cardinals

- Definition: a set is countable if and only if (i) it is $\varnothing$ (ii) or its elements can be listed without omission by an integer-indexed sequence.
- The set of natural numbers 0,1,2 … is countable, since each number can be indexed by itself.
- The set of even numbers is infinite, but countable.

- we can index: 2, 4, 6, 8, 10... by associating any even number 2n with n taken as index.
- There is a function:
  - $f : f(n) = 2n$
- from the set of integers to the set of even numbers.
- This function is surjective, which entails that all even numbers are enumerated, without exception.

Rational numbers are enumerable

- A number is rational iff it can be expressed as a ratio of two integers, in the form m/n.
- 2/3, 3/4, 5/1 (=5) are rational numbers, whose set is .
- How do you enumerate rational numbers?
  - The set a rational numbers is a subset of the set of pairs of integers $\mathbb{N} \times \mathbb{N}$.
- Now, here's a graphical method for enumerating the elements of $\mathbb{N} \times \mathbb{N}$:


1 2 3 4 5 6 7 8 ...
1 1/1 1/2 1/3 1/4 1/5 1/6 1/7 1/8 ...
2 2/1 2/2 2/3 2/4 2/5 2/6 2/7 2/8 ...
3 3/1 3/2 3/3 3/4 3/5 3/6 3/7 3/8 ...
4 4/1 4/2 4/3 4/4 4/5 4/6 4/7 4/8 ...
5 5/1 5/2 5/3 5/4 5/5 5/6 5/7 5/8 ...
6 6/1 6/2 6/3 6/4 6/5 6/6 6/7 6/8 ...
7 7/1 7/2 7/3 7/4 7/5 7/6 7/7 7/8 ...
8 8/1 8/2 8/3 8/4 8/5 8/6 8/7 8/8 ...
⋮ ⋮

$\mathscr{P}(\mathbb{N})$ is not enumerable

- $\mathscr{P}(\mathbb{N})$ is the power set of $\mathbb{N}$, that is, the set of all subsets of $\mathbb{N}$.
- Let us suppose that it is enumerable, and let us imagine that $\boldsymbol{L}$ is any such enumeration:
  - $L = S_1, S_2, \ldots, S_n, \ldots, \ldots$
  - Where each S_i is a subset of $\mathbb{N}$
- Cantor's diagonal argument shows that this list cannot contain all subsets of $\mathbb{N}$.

Cantor's diagonal argument

- Let us define a subset $\bar{D}(L)$ of $\mathbb{N}$ in the following way:
  $\forall n \in \mathbb{N}, n \in \bar{D}(L) \Leftrightarrow n \notin S_n$
  - Where S_n is the n-th element in the list ***L***
- Because of its definition, $\bar{D}(L) \subseteq \mathbb{N}$. So it must be listed in ***L,*** which means that for a certain m, $S_m = \bar{D}(L)$
- Let us apply $\bar{D}(L)$ to m:
  - $\bar{D}(L) : m \in \bar{D}(L) \Leftrightarrow m \notin S_m = \bar{D}(L)$
- So $m \in \bar{D}(L) \Leftrightarrow m \notin \bar{D}(L)$, which a contradiction of the form $p \leftrightarrow \neg p$

- We focus on the diagonal of this matrix, and define an anti-diagonal thus *:*

  - $s(m) = 1 - s_m(m)$

- Then let us imagine that the defined sequence s occurs at line *m,* which means that $s = s_m$

- We still have:

  - $s_m(m) = 1 - s_m(m)$

- Either $s_m(m)$ = 0, or $s_m(m)$ = 1.

- In the first case 0=1.

- In the second case 1=0.

$$
\begin{array}{ll}
s_1 & = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots \\
s_2 & = 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \ldots \\
s_3 & = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ \ldots \\
s_4 & = 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ \ldots \\
s_5 & = 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ \ldots \\
s_6 & = 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ \ldots \\
s_7 & = 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ \ldots \\
s_8 & = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ \ldots \\
s_9 & = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ \ldots \\
s_{10} & = 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ \ldots \\
s_{11} & = 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots \\
\vdots & \quad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
\end{array}
$$

$$
s = 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ \ldots
$$

Infinite cardinals

- For a finite set, its cardinal just is the number of elements that belong to the set.
- But we see that for infinite sets things are more complicated: $\mathbb{N}$ and $\mathscr{P}(\mathbb{N})$ do not have the same cardinal, because they are not equinumerous (= there is no bijection between them).
- So there are different kinds of infinity: countable infinity, corresponding to the cardinal of $\mathbb{N}$, differs from uncountable infinity, which corresponds to the cardinal of $\mathscr{P}(\mathbb{N})$.
- Notations: $Card(\mathbb{N}) = \aleph_0$, $Card(\mathscr{P}(\mathbb{N})) = 2^{\aleph_0}$

IV First-order logic and the infinite

Axioms for « being infinite »

- To what extent is FOL expressive enough to express properties related to infinite cardinals?
- First, let us note that the property « being infinite » can be axiomatized, as we saw in the first part of this course:
  - $\lambda_n = \exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j$
  - Let T be a theory consisting of the infinite set of formulas $\lambda_1, \ldots \lambda_n, \ldots$
  - Obviously, the theory is only satisfied by structures having an infinite domain.
  - Can we define « **being finite** » as well, which would enable us to draw a clear distinction between the finite and the infinite?
    - We can't, because of the compact theorem.

Compactness

- **Compactness** : If all finite subset Γ' of a set Γ have a model, then Γ has a model as well.

- Γ **is satisfiable iff all its finite subsets** Γ' **are satisfiable** (= iff Γ is finitely satisfiable).

Compactness and finiteness

- Let us consider any theory T having arbitrarily large models;
- Let us also consider the sentence :
  - λ_n: $\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j$, for a given number n.
- Since T has arbitrarily large models, $T \vDash \lambda_n$
- Let us consider now $T' = T \cup \{\lambda_n \mid n \in \mathbb{N}\}$; clearly, T' only has infinite models.
- But now, let us turn to any finite subset $T'' \subseteq T'$. We saw that for a given number n, $T \vDash \lambda_n$. So, all models of T are also models of T'', and we have assumed that T has (arbitrarily large) models, hence T" has models.
- So because of compactness, T' also has a model, which: (i) is infinite and (ii) is a model of T.
- Conclusion : **if a theory T has arbitrarily large models, it has a least one infinite model.**

The philosophical importance of compactness

- Compactness suggests that FOL is excellent at characterizing « arbitrarily complicated things about finite patterns » (W. Hodges), but quite bad at distinguishing between structures having different cardinals when infinity is at stake: we can’t distinguish between arbitrarily large structures and infinite structures for instance.

- This is the reason why mathematicians were able to describe **non-standard models for arithmetic or for analysis.**

- The case of non-standard analysis is especially relevant.

Course supplement: Non-standard integers

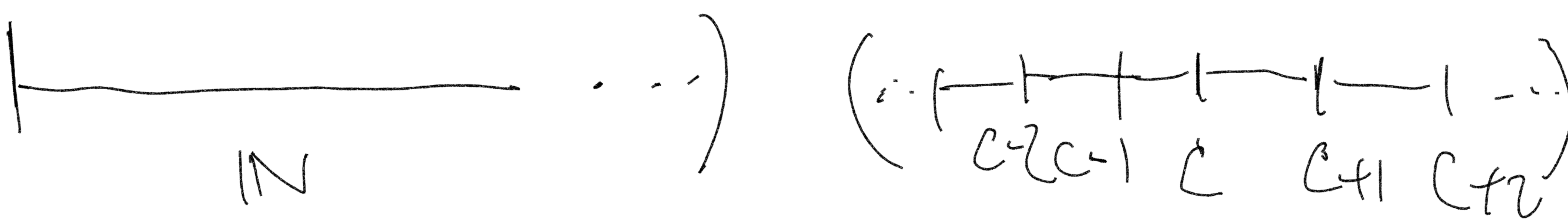
- Let $Th(\mathbb{N})$ be the set of all the sentences true in the standard $\mathscr{L}$-structure $(\mathbb{N}, 0, S, +, <)$, and $\bar{n} = \underbrace{S \ldots S}_{n}(0)$.
- Let us extend the langage of Peano arithmetic with a new constant *c*, and let us consider the set $Th(\mathbb{N}) \cup \{\bar{n} < c \mid n \in \mathbb{N}\}$.
- Now let us take any finite subset of $Th(\mathbb{N}) \cup \{\bar{n} < c \mid n \in \mathbb{N}\}$; it will be included in $T_k = Th(\mathbb{N}) \cup \{\bar{n} < c \mid n < k \in \mathbb{N}\}$.
- Let us consider a model *M* which is identical to the standard model as far as the langage of Peano arithmetic is concerned, and such that I(c) = k.
- $M \vDash T_k$
- So any finite subset of $Th(\mathbb{N}) \cup \{\bar{n} < c \mid n \in \mathbb{N}\}$ has a model ; by compactness: $Th(\mathbb{N}) \cup \{\bar{n} < c \mid n \in \mathbb{N}\}$ has a model, M_c.

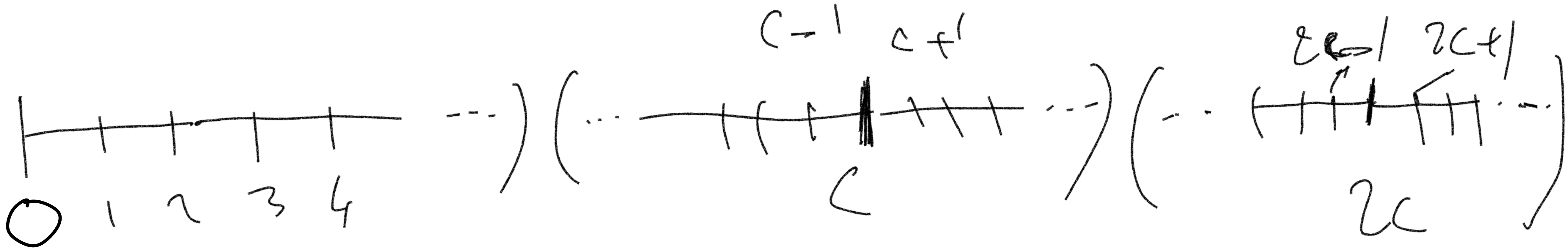
Why is M_c non-standard?

- By construction, $\forall n \in \mathbb{N}, M_c \vDash \bar{n} < c$
- So it contains a non-standard integer, denoted by c, which is infinite in the sense that all standard integers are smaller than it.
- $M_c \vDash \forall x \forall y((x < y) \rightarrow (x \neq y))$
- So M_c and $(\mathbb{N}, 0, S, +, <)$ **are not isomorphic**: there is no integer that could be the inverse of c relative to an isomorphism.

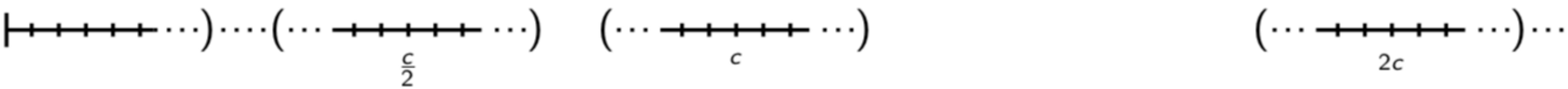
What does it look like?

segment initial de Dc

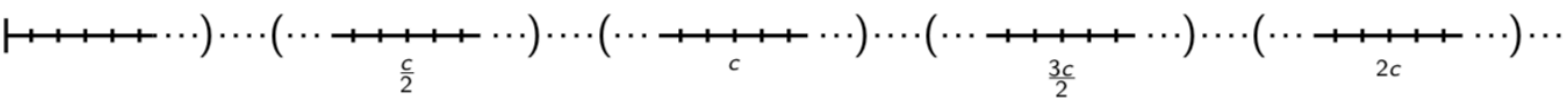




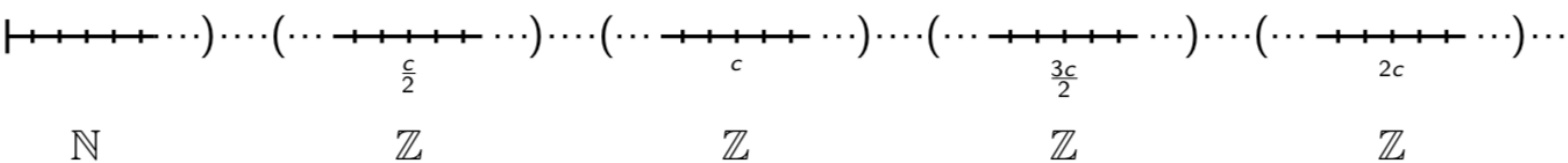
- Let us consider $\frac{c}{2}$. $\forall n, k \in \mathbb{N}, M_c \models \bar{n} < \frac{c}{2} < c - \bar{k}$
- So we get :



By a similar reasoning:



So:



$\mathbb{Q}$ many copies of $\mathbb{Z}$