Solutions to Exercises on vK

Quantified Modal Logic with Variable Domains

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Exercise 1: Prove by tableaux

1.1

Question. Prove by the tableau method in vK:

$$(\Box\forall x\, Px \wedge \Box\forall x\, Qx) \rightarrow \Box\forall x(Px \wedge Qx).$$

Solution. We attempt to close the tableau for $\mathsf{F} : (\Box\forall x\, Px \wedge \Box\forall x\, Qx) \rightarrow \Box\forall x(Px \wedge Qx),\ 1.$

$\mathsf{F} : (\Box\forall x\, Px \land \Box\forall x\, Qx) \to \Box\forall x(Px \land Qx),\ 1$

$\mathsf{T} : \Box\forall x\, Px \land \Box\forall x\, Qx,\ 1$
$\mathsf{F} : \Box\forall x(Px \land Qx),\ 1$

$\mathsf{T} : \Box\forall x\, Px,\ 1$
$\mathsf{T} : \Box\forall x\, Qx,\ 1$ (from $\land$)

$1R2$
$\mathsf{F} : \forall x(Px \land Qx),\ 2$ (fresh 2, from $\mathsf{F}\Box$)

$\mathsf{T} : Ec,\ 2$
$\mathsf{F} : Pc \land Qc,\ 2$ (fresh c, from $\mathsf{F}\forall$)

$\mathsf{T} : \forall x\, Px,\ 2$
$\mathsf{T} : \forall x\, Qx,\ 2$ (from $\mathsf{T}\Box$ using $1R2$)

($\mathsf{T}\forall$ on $\forall x\, Px$ at 2 with c: branch)

$\mathsf{T} : Pc,\ 2$ | $\mathsf{F} : Ec,\ 2\ \times$ (with $\mathsf{T} : Ec$, 2)

($\mathsf{T}\forall$ on $\forall x\, Qx$ at 2 with c: branch)

$\mathsf{T} : Qc,\ 2$ | $\mathsf{F} : Ec,\ 2\ \times$

($\mathsf{F}\land$ on $\mathsf{F} : Pc \land Qc$, 2: branch)

$\mathsf{F} : Qc,\ 2\ \times$ (with $\mathsf{T} : Qc$, 2) | $\mathsf{F} : Pc,\ 2\ \times$ (with $\mathsf{T} : Pc$, 2)

Commentary. The proof combines modal and free-logic quantifier reasoning. The negated conclusion introduces (via $\mathsf{F}\Box$) a fresh world 2 with accessibility $1R2$ and $\mathsf{F} : \forall x(Px \land Qx)$, 2; the $\mathsf{F}\forall$ rule at world 2 then introduces a fresh witness c along with the existence claim $\mathsf{T} : Ec$, 2. The key step is then to apply $\mathsf{T}\Box$ twice at world 1, using the edge $1R2$ to bring both boxed universals down to world 2: $\mathsf{T} : \forall x\, Px$, 2 and $\mathsf{T} : \forall x\, Qx$, 2. Once $\mathsf{T} : Ec$, 2 is on the branch, both $\mathsf{T}\forall$ applications close their left (existence-disjunct) branches immediately. The right branches yield $\mathsf{T} : Pc$, 2 and $\mathsf{T} : Qc$, 2, which together contradict both disjuncts of $\mathsf{F} : Pc \land Qc$, 2. All branches close; the implication is a theorem of vK.

1.2

Question. Prove by the tableau method in vK:

$$\Diamond\exists x\, Px \to \Diamond\exists x(Px \lor Qx).$$

Solution.

$$\mathsf{F} : \Diamond\exists x\, Px \to \Diamond\exists x(Px \lor Qx),\ 1$$

$$\mathsf{T} : \Diamond\exists x\, Px,\ 1$$
$$\mathsf{F} : \Diamond\exists x(Px \lor Qx),\ 1$$

$$1R2$$
$$\mathsf{T} : \exists x\, Px,\ 2 \quad \text{(fresh 2, from } \mathsf{T}\Diamond\text{)}$$

$$\mathsf{T} : Ec,\ 2$$
$$\mathsf{T} : Pc,\ 2 \quad \text{(fresh } c\text{, from } \mathsf{T}\exists\text{)}$$

$$\mathsf{F} : \exists x(Px \lor Qx),\ 2 \quad \text{(from } \mathsf{F}\Diamond \text{ using } 1R2\text{)}$$

$$\text{(}\mathsf{F}\exists \text{ on } c \text{ at world 2: branch)}$$

$$\mathsf{F} : Pc \lor Qc,\ 2 \qquad\qquad \mathsf{F} : Ec,\ 2\ \times \text{ (with } \mathsf{T} : Ec,\ 2\text{)}$$

$$\mathsf{F} : Pc,\ 2,\ \mathsf{F} : Qc,\ 2 \quad \text{(from } \mathsf{F}\lor\text{)}$$

$$\times \text{ (with } \mathsf{T} : Pc,\ 2\text{)}$$

Commentary. The negated conclusion forces $\mathsf{F} : \Diamond\exists x(Px \lor Qx)$, 1, which in turn forces $\mathsf{F} : \exists x(Px \lor Qx)$ at every accessible world (including the world 2 introduced by $\mathsf{T}\Diamond$ on the antecedent). Meanwhile, $\mathsf{T}\exists$ at world 2 supplies a fresh witness c with $\mathsf{T} : Ec$, 2 and $\mathsf{T} : Pc$, 2. The branching $\mathsf{F}\exists$ applied to c at world 2 then closes on both branches: $\mathsf{F} : Ec$, 2 contradicts the freshly-introduced existence claim, and $\mathsf{F} : Pc \lor Qc$, 2 decomposes to $\mathsf{F} : Pc$, 2, which contradicts $\mathsf{T} : Pc$, 2.

1.3

Question. Prove by the tableau method in vK:

$$(\forall x\, \Box Px \land Ea) \to \Box Pa.$$

Solution. This combines the existence-guarded universal instantiation rule with the basic $\Box$ elimination pattern.

$$\mathsf{F} : (\forall x\,\Box Px \wedge Ea) \to \Box Pa,\ 1$$

$$\mathsf{T} : \forall x\,\Box Px \wedge Ea,\ 1$$
$$\mathsf{F} : \Box Pa,\ 1$$

$$\mathsf{T} : \forall x\,\Box Px,\ 1 \qquad \text{(from } \wedge\text{)}$$
$$\mathsf{T} : Ea,\ 1$$

$$1R2 \qquad \text{(fresh 2, from } \mathsf{F}\Box\text{)}$$
$$\mathsf{F} : Pa,\ 2$$

$$(\mathsf{T}\forall \text{ on } \forall x\,\Box Px \text{ at 1 with } a\text{: branch})$$

$$\mathsf{T} : \Box Pa,\ 1 \qquad\qquad \mathsf{F} : Ea,\ 1\ \ \times \text{ (with } \mathsf{T} : Ea,\ 1)$$

$$\mathsf{T} : Pa,\ 2\ \ \text{(from } \mathsf{T}\Box \text{ using } 1R2)$$

$$\times \text{ (with } \mathsf{F} : Pa,\ 2)$$

Commentary. The existence premise $\mathsf{T} : Ea$, 1 is what licenses universal instantiation on the specific constant a. When $\mathsf{T}\forall$ is applied with a, the left branch ($\mathsf{F} : Ea$, 1) closes immediately against the existence premise; the right branch gives us $\mathsf{T} : \Box Pa$, 1. Applying $\mathsf{T}\Box$ using the edge $1R2$ (introduced by the $\mathsf{F}\Box$ rule on the negated conclusion) then yields $\mathsf{T} : Pa$, 2, contradicting $\mathsf{F} : Pa$, 2. Without $\mathsf{T} : Ea$, 1, the left branch would remain open and deliver a counter-model. This is why free-logic universal instantiation, which requires the existence guard, is the correct treatment for vK.

Exercise 2: Non-theorems with counter-models

Show by the tableau method that the following conditionals are not theorems of vK. In each case, construct a counter-model. Does the assumption that the accessibility relation is reflexive change anything?

2.1

Question. Show that the following is not a theorem of vK and construct a counter-model:

$$\Box Pa \to \exists x\,\Box Px.$$

Solution.

Tableau.

$$\mathsf{F} : \Box Pa \to \exists x\, \Box Px,\ 1$$

$$\mathsf{T} : \Box Pa,\ 1$$
$$\mathsf{F} : \exists x\, \Box Px,\ 1$$

$$(\mathsf{F}\exists \text{ on } a \text{ at world 1: branch})$$

$$\mathsf{F} : \Box Pa,\ 1 \ \times \text{(with } \mathsf{T} : \Box Pa,\ 1) \qquad\qquad \mathsf{F} : Ea,\ 1 \ \text{(open—}a \text{ not in } D_1)$$

The left branch is open. We read off the counter-model.

Counter-model.

$$W = \{1\}, \quad R = \emptyset, \quad D = \{\alpha\}, \quad D_1 = \emptyset, \quad I(a) = \alpha.$$

(No assignment to P is needed, since $R = \emptyset$ makes $\Box Pa$ vacuously true.) Verification:

- $\mathcal{M}, 1 \models \Box Pa$: *vacuously*, since no world is accessible from 1. ✓
- $\mathcal{M}, 1 \not\models \exists x\, \Box Px$: the existential ranges over $D_1 = \emptyset$; no witness. ✓

What this shows. The referent of the name a belongs to the outer domain but not to D_1; hence it cannot serve as a witness for an existential evaluated at world 1. A modal claim $\Box Pa$ about a can be true without there being any inner-domain object at 1 that necessarily has P. The failure is a modal variant of the standard existential-generalization failure in free logic.

Does reflexivity change anything? No. Modify the model by adding $1R1$ and setting $I(1, P) = \{\alpha\}$. Then Pa holds at 1 (PFL permits an object that is not in D_1 to fall under P), so $\Box Pa$ still holds at 1. But $D_1 = \emptyset$, so $\exists x\, \Box Px$ still fails. The failure stems from the mismatch between outer and inner domains at the world of evaluation, and the shape of R is powerless to repair it.

2.2

Question. Show that the following is not a theorem of vK and construct a counter-model:

$$\exists x\, \Diamond Px \to \exists x\, \Diamond\Diamond Px.$$

Solution.

Tableau.

$$\mathsf{F} : \exists x \,\Diamond Px \to \exists x \,\Diamond\Diamond Px, \; 1$$

$$\mathsf{T} : \exists x \,\Diamond Px, \; 1$$
$$\mathsf{F} : \exists x \,\Diamond\Diamond Px, \; 1$$

$$\mathsf{T} : Ec, \; 1$$
$$\mathsf{T} : \Diamond Pc, \; 1 \quad \text{(fresh } c\text{, from } \mathsf{T}\exists)$$

$$1R2$$
$$\mathsf{T} : Pc, \; 2 \quad \text{(fresh 2, from } \mathsf{T}\Diamond)$$

($\mathsf{F}\exists$ on c at world 1: branch)

$$\mathsf{F} : \Diamond\Diamond Pc, \; 1 \qquad\qquad \mathsf{F} : Ec, \; 1 \;\; \times \text{ (with } \mathsf{T} : Ec, \, 1)$$

$$\mathsf{F} : \Diamond Pc, \; 2 \;\; \text{(from } \mathsf{F}\Diamond \text{ using } 1R2)$$

(no successor of 2; $\mathsf{F}\Diamond$ at 2 vacuously discharged—open)

The rightmost branch stabilises without closure. We read off the counter-model.

Counter-model.

$$W = \{1, 2\}, \quad 1R2, \quad D = D_1 = D_2 = \{\alpha\}, \quad I(c) = \alpha, \quad I(2, P) = \{\alpha\}, \quad I(1, P) = \emptyset.$$

Verification:

- $\mathcal{M}, 1 \models \exists x \,\Diamond Px$: $\alpha \in D_1$, and $\Diamond Pc$ at 1 since $1R2$ and $\alpha \in I(2, P)$. ✓
- $\mathcal{M}, 1 \not\models \exists x \,\Diamond\Diamond Px$: the only candidate witness is $\alpha \in D_1$. For $\Diamond\Diamond Pc$ at 1, a two-step chain $1RwRv$ with $\alpha \in I(v, P)$ is required. The only successor of 1 is 2, and 2 has no successors; no such chain exists. ✓

What this shows. The antecedent guarantees a witness together with a one-step modal reach to a P-world; the consequent demands a two-step reach from an inner-domain object. Without transitivity or an extra edge, the chain breaks. The principle $\Diamond\varphi \to \Diamond\Diamond\varphi$ corresponds to the *density* condition on R ("every R-edge can be traversed in two steps") and is not forced by vK.

Does reflexivity change anything? Yes. If R is reflexive, the formula becomes valid. Suppose $\exists x \,\Diamond Px$ at w: there is $\alpha \in D_w$ with $\Diamond Pc$ at w, so some v with wRv satisfies Pc at v. By reflexivity vRv, hence $\Diamond Pc$ at v, hence $\Diamond\Diamond Pc$ at w. The same witness $\alpha \in D_w$ serves the consequent, and $\exists x \,\Diamond\Diamond Px$ holds at w. Reflexivity thus repairs the failure: this is not a robustly-invalid formula, but one whose invalidity reflects the absence of a specific condition on R (density, which is guaranteed by reflexivity).

2.3

Question. Show that the following is not a theorem of vK and construct a counter-model:

$$\forall x\, Px \to \exists x\, \Box\Diamond Px.$$

Solution.

Tableau.

$$\mathsf{F} : \forall x\, Px \to \exists x\, \Box\Diamond Px,\ 1$$

$$\mathsf{T} : \forall x\, Px,\ 1$$
$$\mathsf{F} : \exists x\, \Box\Diamond Px,\ 1$$

(no fresh constant introduced; no rule applies to produce one)

(open branch—D_1 empty, no witness available)

The tableau halts without contradiction.

Counter-model.

$$W = \{1\}, \quad R = \emptyset, \quad D = \emptyset, \quad D_1 = \emptyset, \quad I(1, P) = \emptyset.$$

Verification:

- $\mathcal{M}, 1 \models \forall x\, Px$: *vacuously* ($D_1 = \emptyset$). ✓
- $\mathcal{M}, 1 \not\models \exists x\, \Box\Diamond Px$: *vacuously* ($D_1 = \emptyset$)—no witness available. ✓

What this shows. Universal statements are vacuously true at worlds with empty inner domain, while existential statements are vacuously false. The added modal prefix $\Box\Diamond$ in the consequent is irrelevant to the failure: whatever modal operator stands in front of the matrix, the existential requires a witness in D_1, and there is none. The formula therefore fails at any world with empty inner domain.

Does reflexivity change anything? No. Add $1R1$. The formulas remain vacuously true (respectively false) for exactly the same reason—the inner domain is empty, and reflexivity does not populate it.

This is a particularly robust failure. It would persist even under S5-style conditions (reflexivity, symmetry, transitivity), because the source of the failure is the empty-domain phenomenon, not the shape of the accessibility relation.

2.4

Question. Show that the following is not a theorem of vK and construct a counter-model (where E is the existence predicate):

$$\forall x\, \Diamond Ex.$$

Solution.

Tableau.

$$\mathsf{F} : \forall x \, \Diamond Ex, \; 1$$

$$\begin{array}{l} \mathsf{T} : Ec, \; 1 \\ \mathsf{F} : \Diamond Ec, \; 1 \end{array} \quad \text{(fresh } c\text{, from } \mathsf{F}\forall\text{)}$$

(no accessible world from 1; $\mathsf{F}\Diamond$ discharges vacuously—open)

The branch is open. We read off the counter-model.

Counter-model.

$$W = \{1\}, \quad R = \emptyset, \quad D = \{\alpha\}, \quad D_1 = \{\alpha\}, \quad I(c) = \alpha.$$

Verification:

- $\mathcal{M}, 1 \models Ec$: $\alpha \in D_1$. ✓
- $\mathcal{M}, 1 \not\models \Diamond Ec$: no w with $1Rw$; the modal clause is vacuously unsatisfied. ✓
- So $\forall x \, \Diamond Ex$ fails at 1: the inner-domain object α does not possibly-exist at any accessible world.

What this shows. An inner-domain object need not have any modal future at all. In a world with no accessible alternatives, nothing is "possibly F" for any F—including the existence predicate. The formula asks whether every actual object has at least one possibility of existing (including at the actual world, if accessible); vK does not force this.

Does reflexivity change anything? Yes. Under reflexivity, $\forall x \, \Diamond Ex$ becomes a theorem. For any world w and any $\alpha \in D_w$: by reflexivity wRw, and Ec holds at w for the constant c denoting α (given $\alpha \in D_w$); hence $\Diamond Ec$ holds at w. So every inner-domain witness satisfies $\Diamond Ex$ at w, and $\forall x \, \Diamond Ex$ holds.

Equivalently, $Ec \to \Diamond Ec$ follows directly from reflexivity via the T schema applied at the existence predicate. In the counter-model above, adding $1R1$ immediately yields $\Diamond Ec$ at 1 (since Ec holds at 1), and the formula becomes valid.

General observation. The four failures in this exercise divide neatly in two. 2.1 and 2.3 fail for domain-structural reasons (the mismatch between outer and inner domain, or the empty inner domain); reflexivity cannot repair them. 2.2 and 2.4 fail because of the shape of R (absence of density in 2.2, absence of reflexivity in 2.4); adding reflexivity validates them. This illustrates the distinction between structural and relational invalidities in vK.

Exercise 3: vK-NFL with S5-like accessibility

We now work in **vK with negative free logic (NFL)**, where the negativity constraint is imposed: $\langle o_1, \ldots, o_n \rangle \in I(w, P) \Rightarrow o_1, \ldots, o_n \in D_w$ (identity exempt). The accessibility relation is **reflexive, transitive, and symmetric**—that is, an equivalence relation (S5).

3.1

Question. Determine whether the following is a theorem in vK-NFL with S5 accessibility:

$$\exists x\, \Box Px \to \Box \exists x\, Ex.$$

Solution. Theorem of vK-NFL-S5. The result already holds in plain vK-NFL; the S5 conditions on R are not used.

Tableau.

$\mathsf{F} : \exists x\, \Box Px \to \Box\exists x\, Ex,\ 1$

$\mathsf{T} : \exists x\, \Box Px,\ 1$
$\mathsf{F} : \Box\exists x\, Ex,\ 1$

$\mathsf{T} : Ec,\ 1$
$\mathsf{T} : \Box Pc,\ 1$ (fresh c, from $\mathsf{T}\exists$)

$1R2$
$\mathsf{F} : \exists x\, Ex,\ 2$ (fresh 2, from $\mathsf{F}\Box$)

$\mathsf{T} : Pc,\ 2$ (from $\mathsf{T}\Box$ using $1R2$)

$\mathsf{T} : Ec,\ 2$ (NFL: $\mathsf{T} : Pc$, $2 \Rightarrow \mathsf{T} : Ec$, 2)

($\mathsf{F}\exists$ on c at world 2: branch)

$\mathsf{F} : Ec,\ 2\ \times$ (with $\mathsf{T} : Ec$, 2) | $\mathsf{F} : Ec,\ 2\ \times$ (with $\mathsf{T} : Ec$, 2)

All branches close.

Commentary. The crucial step is the NFL *negativity rule* on the true side: any true atomic Pc at a world w licenses $\mathsf{T} : Ec$, w, since NFL forbids P-extensions from containing non-existents. The chain runs as follows. The antecedent supplies a witness c at world 1 with $\mathsf{T} : \Box Pc$, 1. The negated consequent introduces a fresh accessible world 2 with $\mathsf{F} : \exists x\, Ex$, 2. Bringing $\Box Pc$ down to 2 via $\mathsf{T}\Box$ gives $\mathsf{T} : Pc$, 2, and the NFL rule then yields $\mathsf{T} : Ec$, 2. This contradicts both branches of $\mathsf{F}\exists$ applied to c at 2 (the "witness exists" disjunct directly, the "instance" disjunct because the predicate in question is E).

Role of the S5 assumptions. None. Only the single edge $1R2$ produced by $\mathsf{F}\Box$ is used; no appeal is made to reflexivity, transitivity, or symmetry. The theorem holds in plain vK-NFL with any accessibility relation.

Is it a theorem in vK-PFL? No. In PFL, the negativity rule is absent, and $\mathsf{T} : Pc, 2$ no longer licenses $\mathsf{T} : Ec, 2$. The tableau stalls at that step and a counter-model can be constructed: take $W = \{1, 2\}$, $1R2$, $D = \{\alpha\}$, $D_1 = \{\alpha\}$, $D_2 = \emptyset$, $I(c) = \alpha$, $I(1, P) = I(2, P) = \{\alpha\}$. In PFL this is admissible (no negativity constraint), $\Box Pc$ at 1 holds, so $\exists x \, \Box Px$ at 1, but $D_2 = \emptyset$ means $\exists x \, Ex$ fails at 2, so $\Box \exists x \, Ex$ fails at 1. The theorem is specifically an NFL result.

3.2

Question. Determine whether the following is a theorem in vK-NFL with S5 accessibility:

$$\Box \exists x \, Px \to \exists x \, \Box Ex.$$

Solution. Not a theorem of vK-NFL-S5. This is a converse-Barcan-style principle applied to the existence predicate: it says that if every accessible world has some P-object, then some inner-domain object exists throughout all accessible worlds. The witnesses at different accessible worlds can be different objects; no single object need exist everywhere.

Tableau.

$$\mathsf{F} : \Box\exists x\, Px \to \exists x\, \Box Ex,\ 1$$

$$\mathsf{T} : \Box\exists x\, Px,\ 1$$
$$\mathsf{F} : \exists x\, \Box Ex,\ 1$$

$1R1,\ 1R2,\ 1R3$
$2R2,\ 3R3,\ 2R3,\ 3R2$ (S5: reflexive, transitive, symmetric)
$2R1,\ 3R1$

$\mathsf{T} : \exists x\, Px,\ 2$ (from $\mathsf{T}\Box$ using $1R2$)

$\mathsf{T} : Ec_2,\ 2$
$\mathsf{T} : Pc_2,\ 2$ (fresh c_2, from $\mathsf{T}\exists$)

$\mathsf{T} : \exists x\, Px,\ 3$ (from $\mathsf{T}\Box$ using $1R3$)

$\mathsf{T} : Ec_3,\ 3$
$\mathsf{T} : Pc_3,\ 3$ (fresh c_3, from $\mathsf{T}\exists$)

($\mathsf{F}\exists$ on c_2 at 1: branch)

$\mathsf{F} : \Box Ec_2,\ 1$ | $\mathsf{F} : Ec_2,\ 1$ (open: c_2 exists at 2, not at 1)

(yields $\mathsf{F} : Ec_2$ at some accessible world, e.g. 3—open)

The tableau admits open branches. Note that $\mathsf{F}\exists$ must also be applied to c_3 at world 1, with parallel outcomes. The branches stay open because nothing in the tree forces any single constant to exist everywhere. We read off a counter-model.

Counter-model.

$$W = \{1, 2, 3\}, \quad R = W \times W \text{ (equivalence)}, \quad D = \{\alpha, \beta\},$$

$$D_1 = \{\alpha, \beta\}, \quad D_2 = \{\alpha\}, \quad D_3 = \{\beta\},$$
$$I(1, P) = \{\alpha\}, \quad I(2, P) = \{\alpha\}, \quad I(3, P) = \{\beta\}.$$

(The negativity constraint is satisfied: $I(w, P) \subseteq D_w$ at every world.)

Verification:

- $\mathcal{M}, 1 \models \Box\exists x\, Px$: at 1, $\alpha \in D_1$ is P; at 2, $\alpha \in D_2$ is P; at 3, $\beta \in D_3$ is P. ✓
- $\mathcal{M}, 1 \not\models \exists x\, \Box Ex$: candidate witnesses in D_1 are α and β.

– α: $\alpha \notin D_3$, so Ex fails at 3, hence $\Box Ex$ fails at 1 via $1R3$.
– β: $\beta \notin D_2$, so Ex fails at 2, hence $\Box Ex$ fails at 1 via $1R2$.

Neither witness survives; $\exists x\, \Box Ex$ fails at 1. ✓

What this shows. The principle fails for exactly the reason converse Barcan-style formulas fail in variable-domain semantics: $\Box$ binds worlds but does not force *one and the same* witness across those worlds. The antecedent is satisfied by a patchwork of different P-witnesses at different accessible worlds; the consequent demands a single object in the inner domain at 1 that exists at every accessible world, and no such object need exist.

Role of the S5 assumptions. None of reflexivity, transitivity, or symmetry is doing work against the formula here. The counter-model makes R the full equivalence on W, so S5 is satisfied; but the failure would persist in weaker settings (e.g. just $1R2$ and $1R3$). What the S5 setting shows is that even *maximal* accessibility symmetry does not repair the formula: the invalidity is structural (variable domains), not relational.

Is it a theorem in vK-PFL? No, by the same counter-model (PFL imposes fewer constraints than NFL, so any NFL counter-model is automatically a PFL counter-model).

Contrast with 3.1. The asymmetry between 3.1 and 3.2 is revealing. In 3.1, the existential sits *outside* the modal operator in the antecedent: a witness is fixed at the actual world, and the NFL negativity rule propagates the existence of that fixed witness forward via $\Box P$. In 3.2, the existential sits *inside* $\Box$ in the antecedent: each accessible world gets its own local witness, and no single object is forced to persist. The negativity rule cannot bridge this gap, because it only propagates existence along atomic predications of a *given* constant, not across changes of witness.

Summary Table

Formula	**Status in vK**	**Reflexivity helps?**
$(\Box\forall x\, Px \wedge \Box\forall x\, Qx) \to \Box\forall x(Px \wedge Qx)$	Theorem	*n/a—already a theorem*
$\Diamond\exists x\, Px \to \Diamond\exists x(Px \vee Qx)$	Theorem	*n/a*
$(\forall x\, \Box Px \wedge Ea) \to \Box Pa$	Theorem	*n/a*
$\Box Pa \to \exists x\, \Box Px$	Non-theorem	No
$\exists x\, \Diamond Px \to \exists x\, \Diamond\Diamond Px$	Non-theorem	Yes (validates)
$\forall x\, Px \to \exists x\, \Box\Diamond Px$	Non-theorem	No
$\forall x\, \Diamond Ex$	Non-theorem	Yes (validates)
$\exists x\, \Box Px \to \Box\exists x\, Ex$ (NFL-S5)	Theorem	*n/a*
$\Box\exists x\, Px \to \exists x\, \Box Ex$ (NFL-S5)	Non-theorem	No