Quantified Modal Logic with Variable Domains (vK)

Part II of Quantified Modal Logic

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Plan

Recap: where we left off

Last time: cK (constant domains).

- One fixed domain D shared across all worlds.
- All four **Barcan formulas** are theorems.
- **Necessitism** is a theorem: $\forall x \Box Ex$.

But does this match our intuitions?

The contingentist intuition

Had my parents never met, I would not have existed.

Had history gone differently, Sherlock Holmes might have existed.

In general: *what exists seems to be contingent*.

The tense-logical parallel

The same intuition arises even more sharply in the temporal case:

- Socrates existed in 450 BC.
- Socrates does not exist now.
- Your future grandchildren do not yet exist.

If we take tense seriously, the extension of “exists” **varies with the index of evaluation**.

In the modal case, the same thought leads to **variable domains**: different worlds have different domains of quantification.

Variable domain models

Definition

A **variable domain model** is a 5-tuple $\mathcal{M} = \langle W, R, D, \mathrm{dom}, I \rangle$:

- W, R: as before
- D: the **outer domain** (all objects that exist at *some* world)
- $\mathrm{dom} : W \to \mathcal{P}(D)$: assigns to each world w its **inner domain** $D_w \subseteq D$
- I: rigid interpretation of constants (as in cK)

Key innovation: two domains—outer D and per-world inner D_w.

Semantics: quantifiers range over the inner domain

The quantifier clauses change:

$$\mathcal{M}, w, \sigma \models \forall x\, \varphi \quad \Longleftrightarrow \quad \text{for every } o \in D_w,\ \mathcal{M}, w, \sigma[x \mapsto o] \models \varphi$$

$$\mathcal{M}, w, \sigma \models \exists x\, \varphi \quad \Longleftrightarrow \quad \text{for some } o \in D_w,\ \mathcal{M}, w, \sigma[x \mapsto o] \models \varphi$$

Quantifiers now range over D_w—what exists *at the world of evaluation*.

Existence predicate E

$I(w, E) = D_w$. So Ex holds at w iff $\sigma(x) \in D_w$.

Equivalent to $\exists y(x = y)$—with quantifier over inner domain.

Names may refer to non-existents

Rigidity is preserved: a constant c has a fixed referent $I(c) \in D$ (outer domain).

But $I(c)$ may not belong to the inner domain of some world.

Example

At the actual world, "Sherlock Holmes" denotes an object $\sigma \in D$.

But $\sigma \notin D_{\text{actual}}$—Holmes does not exist at our world.

This is what motivates the move to **free logic**.

Why free logic?

Classical rule (existential generalization):

$$\varphi c \ \therefore\ \exists x\, \varphi x$$

This fails in vK.

Counter-example: let c denote an object $o \notin D_w$ (non-existent at w).

- φc may hold at w (object o satisfies the predicate).
- But $\exists x\, \varphi x$ ranges over D_w; if $o \notin D_w$, no witness exists.

Classical logic presupposes *every name denotes in the domain of quantification*. vK abandons this.

$\Rightarrow$ we need **free logic**: explicit existence checks in quantifier rules.

Positive vs. negative free logic

What if c denotes an object not in D_w? Can c satisfy a predicate at w?

Positive free logic (PFL)

Non-existents can satisfy predicates.

E.g.: "Sherlock Holmes is a detective" is true, even though Holmes doesn't exist.

Negative free logic (NFL)

Atomic predication on non-existents is false.

Negativity constraint:
$\langle o_1, \ldots, o_n \rangle \in I(w, P) \Rightarrow o_i \in D_w$
(Identity exempt.)

We treat both systematically.

Tableau rules for vK (PFL)

T∀ (branching on existence)

From T : $\forall x\, \varphi x$, i and constant c: branch into F : Ec, i or T : φc, i.

F∀ (fresh witness)

From F : $\forall x\, \varphi x$, i: introduce fresh c, add both T : Ec, i and F : φc, i.

T∃ (fresh witness)

From T : $\exists x\, \varphi x$, i: introduce fresh c, add both T : Ec, i and T : φc, i.

F∃ (branching on existence)

From F : $\exists x\, \varphi x$, i and constant c: branch into F : Ec, i or F : φc, i.

NFL additional rule

Negativity rule (NFL only)

From T : $Pt_1 \ldots t_n,\ i$, derive T : $Et_k,\ i$ for each argument.

(For atomic predications—*not* for identity.)

Consequence: in NFL, any true atomic predication entails the existence of its arguments.

In PFL, no such rule—non-existents can satisfy predicates.

Order of rule application

The cK procedure carries over, with two amendments for the branching free-logic rules:

- **Apply branching universals last** ($T\forall$, $F\exists$). They split branches; saving them avoids multiplying open branches.
- **Try the existence disjunct first**. When a branching rule fires, check whether the branch already contains $T : Ec$—if so, the existence disjunct closes immediately.

NFL: apply Negativity aggressively—it's non-branching and populates existence facts that help close later branching universals.

Labelled ND rules for vK (PFL)

$\forall$E

From $w : \forall x\, \varphi x$ and $w : Ec$, conclude $w : \varphi c$.

$\forall$I

Assume $w : Ec$ (fresh c); derive $w : \varphi c$; discharge to get $w : \forall x\, \varphi x$.

$\exists$I

From $w : \varphi c$ and $w : Ec$, conclude $w : \exists x\, \varphi x$.

$\exists$E

From $w : \exists x\, \varphi x$, assume $w : \varphi c$, $w : Ec$ (fresh c), derive χ not mentioning c, discharge.

Comparison: cK vs. vK rules

Rule	**cK**	**vK (PFL)**
$\forall$E	$\forall x\, \varphi x \vdash \varphi c$	$\forall x\, \varphi x,\ Ec \vdash \varphi c$
$\forall$I	$\varphi c \vdash \forall x\, \varphi x$ (fresh c)	$[Ec] \vdash \varphi c$, then $\forall x\, \varphi x$
$\exists$I	$\varphi c \vdash \exists x\, \varphi x$	$\varphi c,\ Ec \vdash \exists x\, \varphi x$
$\exists$E	assume φc	assume $\varphi c,\ Ec$

The existence predicate Ec now does real work—it guards every quantifier rule.

Barcan's formulas in vK

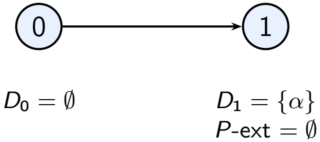
Formula	cK	vK
BF1: $\forall x\, \Box\varphi x \rightarrow \Box\forall x\, \varphi x$	Valid	Invalid
BF2: $\Diamond\exists x\, \varphi x \rightarrow \exists x\, \Diamond\varphi x$	Valid	Invalid
CBF1: $\Box\forall x\, \varphi x \rightarrow \forall x\, \Box\varphi x$	Valid	Invalid
CBF2: $\exists x\, \Diamond\varphi x \rightarrow \Diamond\exists x\, \varphi x$	Valid	Invalid
$\forall x\, \Box Ex$ (necessitism)	Valid	Invalid

All four fail in vK.

Counter-model for BF1

$\forall x \, \Box Px \to \Box \forall x \, Px$:

Model: $W = \{0, 1\}$, $0R1$, $D = \{\alpha\}$, $D_0 = \emptyset$, $D_1 = \{\alpha\}$, $I(a) = \alpha$, $I(1, P) = \emptyset$.



- $\mathcal{M}, 0 \models \forall x \, \Box Px$: *vacuously* ($D_0 = \emptyset$).
- $\mathcal{M}, 0 \not\models \Box \forall x \, Px$: at 1, $\alpha \in D_1$ but $\alpha \notin I(1, P)$.

Counter-model for BF2

$\Diamond \exists x\, Px \to \exists x \Diamond Px$:

Model: $W = \{i, j\}$, iRj, $D = \{\alpha\}$, $D_i = \emptyset$, $D_j = \{\alpha\}$, $I(a) = \alpha$, $I(j, P) = \{\alpha\}$.

- $\mathcal{M}, i \models \Diamond \exists x\, Px$: at j, $\alpha \in D_j$ and $\alpha \in I(j, P)$.
- $\mathcal{M}, i \not\models \exists x \Diamond Px$: $D_i = \emptyset$—no witness at i.

The Pryor scenario, formalised

At an accessible world, Socrates exists and is a philosopher. At our world, Socrates does not exist. So: it is *possible* that something is a philosopher—but there is *nothing* that could be a philosopher.

Necessitism fails in vK

In cK

$\models_{cK} \forall x \Box Ex$

(whatever exists necessarily exists—necessitism is a **theorem**)

In vK

$\not\models_{vK} \forall x \Box Ex$

Counter-model: $W = \{0, 1\}$, $0R1$, $D = \{\alpha\}$, $D_0 = \{\alpha\}$, $D_1 = \emptyset$, $I(a) = \alpha$.

At 0: $\alpha \in D_0$, so Ea holds. But at 1, $\alpha \notin D_1$, so Ea fails. Hence $\Box Ea$ fails at 0.

vK validates the **contingentist** intuition: some existing things might not have existed.

The metaphysical choice

Necessitism

- $\Box\forall x\,\Box\exists y(x = y)$ is valid
- Constant domain
- Barcan formulas are theorems
- cK

Contingentism

- Some things exist contingently
- Variable domains
- Barcan formulas fail
- vK

The choice of logic is a metaphysical commitment.

Neither system is "forced" on us by logic alone.

Williamson's defence of necessitism

Timothy Williamson (2013, *Modal Logic as Metaphysics*): **preserve cK** by reinterpreting.

The distinction

"Exists" $\neq$ "is concretely located".

Socrates *exists* (in the domain of every world), but is *not concretely located* at our world.

Cost: substantial ontological inflation—the domain contains all merely possible objects.

Benefit: preserves the simplicity of cK; matches the treatment of abstract objects.

Possibilism vs. actualism

A related but distinct dispute:

Possibilism

- There *are* mere possibilia
- Wittgenstein's possible child, Sherlock Holmes, ...
- Outer domain D holds them

Actualism

- All there is, is all there *actually* is
- No mere possibilia
- Outer domain = inner domain of @

Not the same as necessitism vs. contingentism:

Williamson is a necessitist without being a (traditional) possibilist—his merely possible objects are *actual* (as abstract entities), not ghostly possibilia.

The tense-logical parallel

Eternalism

- All times are ontologically on a par
- Socrates, dinosaurs, future grandchildren all exist—just at other temporal regions
- Tense logic with constant domain
- Barcan formulas are theorems

Presentism

- Only the present is real
- Socrates did exist but doesn't anymore
- Tense logic with variable domains
- Barcan formulas fail

Same structural issue, different dimension.

The deep issue is the ontological status of *non-located objects*.

The asymmetry of identity

In cK, identity rules are bookkeeping—the constant domain does the work.

In vK, the *same rules* become substantive commitments:

- **Rigidity through non-existence**: $a = a$ holds at worlds where a's referent does not exist in the inner domain.
- **Necessity of identity at non-existents**: $\Box(a = b)$ holds at every world, including worlds where a, b do not exist.
- **Identity exempt from NFL negativity**: assumes identity is a *logical* predicate, not a contingent one.

Alternative systems are coherent: neutral free logic, contingent identity (Gibbard 1975), counterpart theory (Lewis 1968).

Summary

- **vK** = K + free first-order logic, with variable domains.
- Two domains: outer D, per-world inner D_w.
- Constants are still **rigid**—but may denote objects outside the inner domain.
- Quantifier rules **guard by existence**; Barcan formulas all **fail**.
- Two free logics: **PFL** (atomic predications on non-existents allowed) vs. **NFL** (forbidden).
- Metaphysical stakes: **contingentism**, **possibilism/actualism**, **eternalism/presentism**.

The moral

Logic and metaphysics are interleaved. Choice of formal system is a choice of metaphysical picture.