Solutions to Exercises

Lecture Notes on Quantified Modal Logic

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Contents

This document provides detailed solutions to all exercises in the lecture notes on quantified modal logic. Exercises are organised by the Part in which they appear. Each solution restates the exercise, gives a proof, derivation, or counter-model, and comments briefly on the result where appropriate. The numbering of sections follows the Part structure of the comprehensive notes.

Where a question is technical (a tableau or labelled natural deduction proof), we give the proof in full, together with an explanation of the key steps. Where a question is philosophical (typically in Part IV), we give a structured argument that makes the main considerations explicit; these are not the only defensible answers, but they illustrate the kind of response expected.

Readers should attempt each exercise before consulting the solution. The point of the exercise is the struggle, not the answer.

Part I

Solutions to Exercises from Part I (cK)

This part covers the nine exercises from the constant-domain sections of the notes: three exercises on identity (Section 5 of the notes) and six exercises at the end of Part I (Section 8 of the notes).

1 Identity Exercises

Solution 1.1 (Exercise 5.1: $\forall x(x = x)$ in cK)**. Claim:** $\vdash_{cK} \forall x(x = x)$.

Proof. At any world w, and for any fresh constant c, Reflexivity gives $w : c = c$; $\forall$I then generalises:

1	fresh c	
2	$w : c = c$	Reflexivity
3	$w : \forall x(x = x)$	$\forall$I, 1—2

Remark. Reflexivity of identity is the most basic identity principle and is available at every world. The $\forall$I step requires that c be fresh, which is trivial here since c does not appear outside the subproof.

Solution 1.2 (Exercise 5.2: Modal Leibniz law)**. Claim:** $\vdash_{cK} \forall x \forall y(x = y \to (Px \to Py))$.

Tableau. We show the tableau for the negation closes.

$$\mathsf{F} : \forall x \forall y(x = y \to (Px \to Py)),\ 1$$

$$\mathsf{F} : \forall y(a = y \to (Pa \to Py)),\ 1 \quad \text{(fresh } a\text{)}$$

$$\mathsf{F} : a = b \to (Pa \to Pb),\ 1 \quad \text{(fresh } b\text{)}$$

$$\mathsf{T} : a = b,\ 1$$

$$\mathsf{F} : Pa \to Pb,\ 1$$

$$\mathsf{T} : Pa,\ 1,\ \mathsf{F} : Pb,\ 1$$

$$\mathsf{T} : Pb,\ 1 \quad \text{(Substitution from } \mathsf{T} : a = b, 1 \text{ and } \mathsf{T} : Pa, 1\text{)}$$

$$\times$$

Remark. The proof does not use the Rigidity rule; everything takes place at the single world 1. The modal Leibniz law is a theorem of cK even at a single world, because identity together with Substitution already give us indiscernibility.

Solution 1.3 (Exercise 5.3: Necessary substitutivity of identicals)**. Claim:** $\vdash_{cK} \forall x \forall y(x = y \to \Box\forall z(z = x \leftrightarrow z = y))$.

Proof in labelled ND.

1		fresh a, b
2	$w : a = b$	H
3	wRv	fresh v
4	$v : a = b$	Rigidity, 2
5		fresh c
6	$v : c = a$	H
7	$v : c = b$	Substitution, 6, 4
8	$v : c = a \to c = b$	$\to$I, 6—7
9	$v : c = b$	H
10	$v : c = a$	Substitution, 9, 4
11	$v : c = b \to c = a$	$\to$I, 9—10
12	$v : c = a \leftrightarrow c = b$	$\leftrightarrow$I, 8, 11
13	$v : \forall z(z = a \leftrightarrow z = b)$	$\forall$I, 5—12
14	$w : \Box\forall z(z = a \leftrightarrow z = b)$	$\Box$I, 3—13
15	$w : a = b \to \Box\forall z(z = a \leftrightarrow z = b)$	$\to$I, 2—14
16	$w : \forall x \forall y(x = y \to \Box\forall z(z = x \leftrightarrow z = y))$	$\forall$I, 1—15

Remark. The proof relies on three things: the Rigidity rule (line 4), Substitution of identicals (lines 7, 10), and the standard ND rules for $\to$I, $\forall$I, and $\Box$I. The structure is straightforward once one sees that the entire left-to-right and right-to-left halves of the biconditional are symmetric and both follow from Substitution.

2 General cK Exercises

Solution 2.1 (Exercise 9.1: Necessity-commutes-with-universal)**. Claim:** $\forall x\, \Box\varphi x \vdash_{cK} \Box\forall x\, \varphi x$.

Tableau. We show the tableau with both the premise and the negation of the conclusion closes.

$$\mathsf{T} : \forall x\, \Box \varphi x,\ 1$$
$$\mathsf{F} : \Box \forall x\, \varphi x,\ 1$$

$1R2$
$\mathsf{F} : \forall x\, \varphi x,\ 2$ (fresh 2, from $\mathsf{F}\Box$)

$\mathsf{F} : \varphi c,\ 2$ (fresh c, from $\mathsf{F}\forall$)

$\mathsf{T} : \Box \varphi c,\ 1$ (from $\mathsf{T}\forall$ with c)

$\mathsf{T} : \varphi c,\ 2$ (from $\mathsf{T}\Box$ using $1R2$)

$\times$

Labelled ND. The same result:

1	$w : \forall x\, \Box \varphi x$	P
2	wRv	fresh v
3		fresh c
4	$w : \Box \varphi c$	$\forall$E, 1
5	$v : \varphi c$	$\Box$E, 4, 2
6	$v : \forall x\, \varphi x$	$\forall$I, 3—5
7	$w : \Box \forall x\, \varphi x$	$\Box$I, 2—6

Remark. This is BF1 with the implication unpacked into a sequent. The proof uses CBF1's deep structure: $\forall$E on the $\Box$-boxed universal, then $\Box$E through the same accessibility edge, then $\forall$I on the arbitrary c, then $\Box$I closing the accessibility subproof.

Solution 2.2 (Exercise 9.2: Quantified K axiom)**. Claim:** $\vdash_{cK} \Box \forall x(Px \to Qx) \to (\Box \forall x\, Px \to \Box \forall x\, Qx)$.

Proof in labelled ND.

1	$w : \Box\forall x(Px \to Qx)$	H
2	$w : \Box\forall x\, Px$	H
3	wRv	fresh v
4	$v : \forall x(Px \to Qx)$	$\Box$E, 1, 3
5	$v : \forall x\, Px$	$\Box$E, 2, 3
6		fresh c
7	$v : Pc \to Qc$	$\forall$E, 4
8	$v : Pc$	$\forall$E, 5
9	$v : Qc$	$\to$E, 7, 8
10	$v : \forall x\, Qx$	$\forall$I, 6—9
11	$w : \Box\forall x\, Qx$	$\Box$I, 3—10
12	$w : \Box\forall x\, Px \to \Box\forall x\, Qx$	$\to$I, 2—11
13	$w : \Box\forall x(Px \to Qx) \to (\Box\forall x\, Px \to \Box\forall x\, Qx)$	$\to$I, 1—12

Remark. The proof nests two $\to$I steps around a single $\Box$I subproof. Inside the $\Box$I subproof, both boxed universals release their content (lines 4 and 5), which is then combined pointwise by $\forall$E / modus ponens / $\forall$I.

Solution 2.3 (Exercise 9.3: Existential possibility distribution)**. Claim:** $\Diamond\exists x(Px \wedge Qx) \vdash_{cK} \Diamond\exists x\, Px \wedge \Diamond\exists x\, Qx$. The converse is *not* derivable.

Proof of the forward direction.

1	$w : \Diamond\exists x(Px \wedge Qx)$	P
2	$wRv,\ v : \exists x(Px \wedge Qx)$	fresh v
3	$v : Pc \wedge Qc$	fresh c
4	$v : Pc$	$\wedge$E, 3
5	$v : Qc$	$\wedge$E, 3
6	$v : \exists x\, Px$	$\exists$I, 4
7	$v : \exists x\, Qx$	$\exists$I, 5
8	$w : \Diamond\exists x\, Px$	$\Diamond$I, 2, 6
9	$w : \Diamond\exists x\, Qx$	$\Diamond$I, 2, 7
10	$w : \Diamond\exists x\, Px \wedge \Diamond\exists x\, Qx$	$\wedge$I, 8, 9
11	$w : \Diamond\exists x\, Px \wedge \Diamond\exists x\, Qx$	$\exists$E, 2, 3—10
12	$w : \Diamond\exists x\, Px \wedge \Diamond\exists x\, Qx$	$\Diamond$E, 1, 2—11

The converse fails. Consider the counter-model $W = \{w_1, w_2, w_3\}$, $w_1 R w_2$, $w_1 R w_3$, $D = \{\alpha\}$, with $I(w_2, P) = \{\alpha\}$, $I(w_2, Q) = \emptyset$, $I(w_3, P) = \emptyset$, $I(w_3, Q) = \{\alpha\}$. Then

- $\mathcal{M}, w_1 \models \Diamond\exists x\, Px$ (witnessed by w_2);
- $\mathcal{M}, w_1 \models \Diamond\exists x\, Qx$ (witnessed by w_3);
- $\mathcal{M}, w_1 \not\models \Diamond\exists x(Px \wedge Qx)$: at neither accessible world is any object both P and Q.

Remark. The converse is the "conjunctive distribution" of $\Diamond$; it fails because $\Diamond$ does not commute with $\wedge$ over different accessible worlds. The two conjuncts may be satisfied at different worlds but not at the same one.

Solution 2.4 (Exercise 9.4: De dicto / de re gap)**. Claim:** $\Box\exists x(Px) \nvdash_{cK} \exists x\,\Box Px$.

Tableau. We attempt to close a tableau with $\mathsf{T} : \Box\exists x\, Px$, 1 and $\mathsf{F} : \exists x\,\Box Px$, 1.

$\mathsf{T} : \Box\exists x\, Px,\ 1$
$\mathsf{F} : \exists x\,\Box Px,\ 1$

$1R2,\ \mathsf{T} : \exists x\, Px,\ 2$ (creating w_2 via $\mathsf{T}\Box$)

$\mathsf{T} : Pa,\ 2$ (fresh a, via $\mathsf{T}\exists$)

$\mathsf{F} : \Box Pa,\ 1$ ($\mathsf{F}\exists$ with a)

$1R3,\ \mathsf{F} : Pa,\ 3$ (fresh 3, from $\mathsf{F}\Box$)

$\mathsf{T} : \exists x\, Px,\ 3$ (from $\mathsf{T}\Box$ using $1R3$)

$\mathsf{T} : Pb,\ 3$ (fresh b, via $\mathsf{T}\exists$)

(open branch; no closure possible—$b \neq a$)

Counter-model read off. $W = \{1, 2, 3\}$, $1R2$, $1R3$, $D = \{\alpha, \beta\}$, $I(a) = \alpha$, $I(b) = \beta$, $I(2, P) = \{\alpha\}$, $I(3, P) = \{\beta\}$. Then:

- $\mathcal{M}, 1 \models \Box\exists x\, Px$: at every accessible world, *something* is P.
- $\mathcal{M}, 1 \not\models \exists x\,\Box Px$: neither α nor β is P at *every* accessible world.

Remark. The failure is exactly the de dicto / de re gap: it can be necessary that *something* is P (de dicto) without there being anything that is necessarily P (de re). Which object plays the "something" role may vary from world to world.

Solution 2.5 (Exercise 9.5: Necessary biconditional for identicals)**. Claim:** $\vdash_{cK} a = b \rightarrow \Box(Pa \leftrightarrow Pb)$.

Proof in labelled ND.

1	$w : a = b$	H
2	wRv	fresh v
3	$v : a = b$	Rigidity, 1
4	$v : Pa$	H
5	$v : Pb$	Substitution, 4, 3
6	$v : Pa \to Pb$	$\to$I, 4—5
7	$v : Pb$	H
8	$v : Pa$	Substitution, 7, 3
9	$v : Pb \to Pa$	$\to$I, 7—8
10	$v : Pa \leftrightarrow Pb$	$\leftrightarrow$I, 6, 9
11	$w : \Box(Pa \leftrightarrow Pb)$	$\Box$I, 2—10
12	$w : a = b \to \Box(Pa \leftrightarrow Pb)$	$\to$I, 1—11

Remark. Rigidity (line 3) is the essential step. Once the identity is carried to the accessible world v, Substitution gives us both directions of the biconditional.

Solution 2.6 (Exercise 9.6: $\Box Pa \not\models \forall x \Box Px$)**. Claim:** A counter-model showing that de dicto $\Box Pa$ does not imply de re $\forall x \Box Px$.

Counter-model. Take $W = \{w_1, w_2\}$, $w_1 R w_2$, $D = \{\alpha, \beta\}$, $I(a) = \alpha$, with $I(w_1, P) = \{\alpha\}$, $I(w_2, P) = \{\alpha\}$.

- $\mathcal{M}, w_1 \models \Box Pa$: at w_2 (the only accessible world), $\alpha \in I(w_2, P)$.
- $\mathcal{M}, w_1 \not\models \forall x \Box Px$: take $\beta \in D$; then $\mathcal{M}, w_1, \sigma[x \mapsto \beta] \not\models \Box Px$, because at w_2, $\beta \notin I(w_2, P)$.

What this shows. $\Box Pa$ is a de dicto claim: a is necessarily P, as a statement about the specific name a. But this does not generalise: there may be other objects in the domain about which no necessity claim holds. The universal $\forall x \Box Px$ requires that *every* object be necessarily P; the single case a does not settle that. This is a version of the general warning that de dicto necessity (about a specific proposition) and universal de re necessity (about every object) are distinct: instance-by-instance necessity does not aggregate into universal necessity.

Part II

Solutions to Exercises from Part II (vK)

Seven exercises on variable-domain quantified modal logic, covering both positive and negative free logic.

3 vK Exercises

Solution 3.1 (Exercise 17.1: Existence-restricted instantiation)**. Claim:** $\forall x\, Px,\ \exists x(x = a) \vdash_{vK\text{-PFL}} Pa$.

Proof in labelled ND.

1	$w : \forall x\, Px$	P
2	$w : \exists x(x = a)$	P
3	$\quad w : c = a,\ w : Ec$	fresh c
4	$\quad w : Pc$	$\forall$E, 1, 3
5	$\quad w : Pa$	Substitution, 4, 3
6	$w : Pa$	$\exists$E, 2, 3—5

Remark. The hypothesis $\exists x(x = a)$ plays the role of an existence guard: it licenses the existential elimination that gives us a constant c with $c = a$ *and* Ec. With Ec in hand, $\forall$E applies, and Substitution substitutes a for c.

Solution 3.2 (Exercise 17.2: $\exists y(y = a) \vdash Ea$)**. Claim:** $\exists y(y = a) \vdash_{vK} Ea$.

Tableau. We show the tableau with the premise and $\mathsf{F} : Ea$, 1 closes.

$$\mathsf{T} : \exists y(y = a),\ 1$$
$$\mathsf{F} : Ea,\ 1$$

$$\mathsf{T} : Ec,\ 1$$
$$\mathsf{T} : c = a,\ 1$$
(fresh c, from $\mathsf{T}\exists$)

$\mathsf{T} : Ea,\ 1$ (Substitution from $\mathsf{T} : c = a$, 1 and $\mathsf{T} : Ec$, 1)

$$\times$$

Remark. This is what justifies treating Ex and $\exists y(x = y)$ as equivalent throughout Part II. The converse direction ($Ea \vdash \exists y(y = a)$) uses Reflexivity and $\exists$I: from $a = a$ and Ea, $\exists$I gives $\exists y(y = a)$.

Solution 3.3 (Exercise 17.3: Universal implies distribute)**. Claim:** $\forall x(Px \to Qx) \vdash_{vK\text{-PFL}} \forall x\, Px \to \forall x\, Qx$.

Proof.

1	$w : \forall x(Px \to Qx)$	P
2	$\quad w : \forall x\, Px$	H
3	$\quad\quad w : Ec$	H (fresh c)
4	$\quad\quad w : Pc \to Qc$	$\forall$E, 1, 3
5	$\quad\quad w : Pc$	$\forall$E, 2, 3
6	$\quad\quad w : Qc$	$\to$E, 4, 5
7	$\quad w : \forall x\, Qx$	$\forall$I, 3—6
8	$w : \forall x\, Px \to \forall x\, Qx$	$\to$I, 2—7

Remark. The existence assumption Ec is introduced at line 3 in the $\forall$I subproof. It is required for both applications of $\forall$E (lines 4 and 5) and is discharged by $\forall$I at line 7. The proof mirrors the cK version but with the explicit existence guard that vK demands.

Solution 3.4 (Exercise 17.4: $\forall x(Px \to Ex)$ in NFL)**. Claim in NFL:** $\vdash_{vK\text{-NFL}} \forall x(Px \to Ex)$.

Proof.

1	$\quad w : Ec$	H (fresh c)
2	$\quad\quad w : Pc$	H
3	$\quad\quad w : Ec$	Negativity (NFL), 2
4	$\quad w : Pc \to Ec$	$\to$I, 2—3
5	$w : \forall x(Px \to Ex)$	$\forall$I, 1—4

Line 3 is strictly speaking redundant given line 1, but we prefer to show Negativity in action: from Pc, we derive Ec at the same world. $\to$I then discharges the Pc assumption; $\forall$I discharges the Ec assumption at the outer layer.

Is it a theorem of PFL? *No.* In PFL, Pc does not entail Ec: the positive free logician allows non-existents to satisfy predicates (the Sherlock Holmes case). A counter-model in PFL: $W = \{w\}$, $D_w = \emptyset$, $I(a) = \alpha$, $I(w, P) = \{\alpha\}$. Then Pa holds at w but $\alpha \notin D_w$, so Ea fails and the universal fails.

Remark. This exercise illustrates a concrete difference between PFL and NFL: the latter validates the schema "atomic predication entails existence", the former does not.

Solution 3.5 (Exercise 17.5: vK-PFL counter-model for $Pa \to \exists x\, Px$)**. Claim:** $\not\models_{vK\text{-PFL}} Pa \to \exists x\, Px$.

Counter-model. Take:

- $W = \{w\}$, $R = \emptyset$;
- outer domain $D = \{\alpha\}$;
- inner domain $D_w = \emptyset$;
- $I(a) = \alpha$;
- $I(w, P) = \{\alpha\}$.

Verification.

- $\mathcal{M}, w \models Pa$: $I(a) = \alpha \in I(w, P)$. So Pa holds at w. (This is legitimate in PFL because the negativity constraint is not imposed.)
- $\mathcal{M}, w \not\models \exists x\, Px$: the existential ranges over $D_w = \emptyset$, so no witness exists.

Hence $\mathcal{M}, w \not\models Pa \to \exists x\, Px$.

Remark. This is the paradigm case motivating free logic. In classical first-order logic, Pa always licences $\exists x\, Px$. In vK-PFL with a non-denoting name, the implication fails: the name refers to an object *in the outer domain*, but the existential quantifier ranges only over *the inner domain*, which may exclude the object.

Solution 3.6 (Exercise 17.6: $Pa \vdash \exists x\, Px$ in NFL)**. Claim in NFL:** $Pa \vdash_{vK\text{-NFL}} \exists x\, Px$.

Proof.

1	$w : Pa$	P
2	$w : Ea$	Negativity (NFL), 1
3	$w : \exists x\, Px$	$\exists$I, 1, 2

Contrast with PFL. In vK-PFL, this derivation fails because there is no Negativity rule: line 2 cannot be obtained from line 1. Without Ea, $\exists$I at line 3 cannot fire. The counter-model from the previous exercise witnesses this: in it, Pa holds but $\exists x\, Px$ does not.

Remark. NFL restores existential generalization for atomic predications. PFL does not. This is exactly the point of distinguishing the two systems: positive and negative free logic give different answers to the question of whether atomic predication on a non-existent is admissible.

Solution 3.7 (Exercise 17.7: $\forall x\, \Box Ex$ in vK)**. Claim:** $\forall x\, \Box Ex$ is *not* vK-valid.

Counter-model. $W = \{w_1, w_2\}$, $w_1 R w_2$, outer domain $D = \{\alpha\}$, $D_{w_1} = \{\alpha\}$, $D_{w_2} = \emptyset$, $I(a) = \alpha$.

- At w_1, the inner domain contains α, so Ea holds at w_1. But $\Box Ea$ requires Ea at every accessible world. At w_2, $\alpha \notin D_{w_2}$, so Ea fails at w_2.
- Hence $\mathcal{M}, w_1 \not\models \forall x\, \Box Ex$: the single object $\alpha \in D_{w_1}$ does not necessarily exist.

Comparison with cK. In cK, $\forall x\, \Box Ex$ *is* valid (Theorem the Necessary-Existence Theorem of the lecture notes was cited to this effect). This is because the domain is constant: every object exists at every world. In vK, the very point of variable domains is to allow an object to exist at one world and fail to exist at another; so $\forall x\, \Box Ex$ is the formal statement of necessitism, and vK rejects it exactly because vK allows contingent existence.

Remark. This is the core divergence between cK and vK, formalised. Which system one adopts depends on whether one takes existence to be a contingent or a necessary matter.

Part III

Solutions to Exercises from Part III (Barcan Failures)

4 Barcan Failure Exercises

Solution 4.1 (Exercise 23.1: $\Box Ea \to Ea$ fails in vK)**. Claim:** $\not\models_{vK} \Box Ea \to Ea$.

Counter-model. Take $W = \{w_1, w_2\}$, $w_1 R w_2$, $D = \{\alpha\}$, $D_{w_1} = \emptyset$, $D_{w_2} = \{\alpha\}$, $I(a) = \alpha$.

- $\mathcal{M}, w_1 \models \Box Ea$: at w_2 (the only accessible world), $\alpha \in D_{w_2}$, so Ea holds at w_2. Hence $\Box Ea$ holds at w_1.
- $\mathcal{M}, w_1 \not\models Ea$: $\alpha \notin D_{w_1}$, so Ea fails at w_1.

Hence $\mathcal{M}, w_1 \not\models \Box Ea \to Ea$.

What the counter-model reveals. The formula $\Box Ea \to Ea$ is the modal-logical analogue of the axiom schema T ($\Box\varphi \to \varphi$), which is valid only in reflexive frames. In vK, the interesting phenomenon is that Ea can fail at one world and hold at another, regardless of the accessibility structure. In our model, a's referent does not exist at w_1 but does exist at w_2; the accessibility from w_1 to w_2 allows $\Box Ea$ to be true at w_1, even though Ea itself is false there. The counter-model shows that vK takes seriously the possibility that objects might come into existence at accessible worlds without existing at the world from which we evaluate modal claims.

Solution 4.2 (Exercise 23.2: CBF2 fails in vK)**. Claim:** $\not\models_{vK} \exists x \Diamond \varphi x \to \Diamond \exists x \varphi x$ (CBF2).

Counter-model. $W = \{w_1, w_2\}$, $w_1 R w_2$, $D = \{\alpha\}$, $D_{w_1} = \{\alpha\}$, $D_{w_2} = \emptyset$, $I(a) = \alpha$, $I(w_2, P) = \{\alpha\}$ (note: $\alpha \notin D_{w_2}$ yet is in the extension of P at w_2—this requires PFL).

Verification.

- $\mathcal{M}, w_1 \models \exists x \Diamond Px$: take $\alpha \in D_{w_1}$. For $\mathcal{M}, w_1, \sigma[x \mapsto \alpha] \models \Diamond Px$, we need some accessible world where Px holds. At w_2, $\alpha \in I(w_2, P)$, so Px holds under the assignment. Hence $\Diamond Px$ holds of α at w_1, witnessing the existential.
- $\mathcal{M}, w_1 \not\models \Diamond \exists x Px$: for this, we would need some accessible world where $\exists x Px$ holds. At w_2, the inner domain is empty, so $\exists x Px$ fails there. There is no other accessible world. So $\Diamond \exists x Px$ fails at w_1.

Hence $\mathcal{M}, w_1 \models \exists x \Diamond Px$ but $\mathcal{M}, w_1 \not\models \Diamond \exists x Px$; CBF2 fails.

Remark. This counter-model is a pure PFL construction: we require the extension of P at w_2 to contain α even though $\alpha \notin D_{w_2}$. In NFL, the negativity constraint would forbid this, so the model is illegal in NFL. CBF2 does, however, still fail in NFL—a different counter-model exhibits the failure there.

Solution 4.3 (Exercise 23.3: Free-logic universal instantiation)**. Claim:** $\models_{vK} \forall x Px \to (Pa \vee \neg Ea)$.

Proof. Suppose $\mathcal{M}, w \models \forall x Px$. Two cases.

- If $\mathcal{M}, w \models Ea$, then $I(a) \in D_w$. Since $\forall x Px$ holds at w, Px holds of every object in D_w, including $I(a)$. So Pa holds at w. Hence $Pa \vee \neg Ea$ holds.
- If $\mathcal{M}, w \not\models Ea$, then $\neg Ea$ holds. So $Pa \vee \neg Ea$ holds.

In either case, $Pa \vee \neg Ea$ holds at w.

Labelled ND proof.

1	$w : \forall x\, Px$	H
2	$w : Ea$	H
3	$w : Pa$	$\forall$E, 1, 2
4	$w : Pa \lor \neg Ea$	$\lor$I, 3
5	$w : Ea \to Pa \lor \neg Ea$	$\to$I, 2—4
6	$w : \neg Ea$	H
7	$w : Pa \lor \neg Ea$	$\lor$I, 6
8	$w : \neg Ea \to Pa \lor \neg Ea$	$\to$I, 6—7
9	$w : Ea \lor \neg Ea$	LEM
10	$w : Pa \lor \neg Ea$	$\lor$E, 9, 5, 8
11	$w : \forall x\, Px \to (Pa \lor \neg Ea)$	$\to$I, 1—10

Remark. In classical FOL, the valid schema is $\forall x\, Px \to Pa$. In free logic, this fails (a name may not denote in the inner domain). The valid replacement is $\forall x\, Px \to (Pa \lor \neg Ea)$: if everything is P, then either a is P, or a doesn't exist.

Part IV

Solutions to Exercises from Part VI (Philosophy)

These exercises are philosophical rather than technical. Our solutions give a structured argument making the main considerations explicit; alternative defensible positions are possible.

5 Philosophical Exercises

Solution 5.1 (Exercise 32.1: Evaluating Williamson's distinction)**. Question.** Evaluate Williamson's distinction between "existing" and "being concretely located". Does it succeed in reconciling our intuitions about contingent existence with the simplicity of cK? What costs does it incur?

Discussion. Williamson's strategy is to preserve cK (and thus necessitism) by reinterpreting what "exists" means. On his view, Socrates exists simpliciter—he is in the domain of every world—but he is not *concretely located* at our world. The cost of contingent existence is shifted from being to concreteness.

Two strengths.

1. It preserves the clean formal framework of cK, including the Barcan formulas. No free logic is required, no variable domains, no existence predicate in the substantive sense.

2. It respects a common intuition about abstract existents. Numbers, properties, sets, etc., seem to "exist" in some sense without being concretely located anywhere. If we accept that abstract objects have this kind of uncommitted existence, then extending the same treatment to "contingently non-located" particulars is a natural move.

Two costs.

1. *Ontological inflation.* Williamson's domain contains merely possible objects: every person who could have existed, every object of any possible kind. The outer reaches of this ontology are vast. Many philosophers (Quine most notably) find this profligate.

2. *The intuition dodge.* The ordinary thought "I might not have existed" is rendered as "I might have been non-concrete". But this is not what we ordinarily mean. We mean: I might simply have not been there at all, not that I might have existed as an abstract fact. Williamson's translation may be formally coherent but seems to miss the phenomenology of the contingentist intuition.

Overall evaluation. Williamson's position is formally elegant and coherent. Whether it successfully reconciles our intuitions with cK depends on how much weight one places on the costs. Those who find ontological inflation a serious objection will reject it; those who prize formal simplicity and believe abstract objects are respectable citizens of the ontology will accept it. The dispute is genuinely metaphysical, not merely logical.

Solution 5.2 (Exercise 32.2: cK without necessitism?)**. Question.** Is there a coherent position that accepts the formal apparatus of cK (constant domains, Barcan formulas) while rejecting the metaphysical doctrine of necessitism?

Discussion. The question turns on what one takes the interpretation of the formal apparatus to be. If the constant domain D of cK is interpreted as "the set of all things that exist (in the maximally inclusive sense)", then validating $\forall x\,\Box Ex$ (the Barcan-style necessitist theorem) is just to affirm that everything in the inclusive sense necessarily exists in the inclusive sense—which is trivial.

However, one could reinterpret the domain D as something else. For example:

- *Possible objects* (possibilism): D is the set of all possibilia, and Ex is read as "x is actual". Under this reading, the formula $\forall x\,\Box Ex$ is not a claim of necessitism but a claim that all possibilia are actual—which, in the possibilist picture, may be rejected on independent grounds.
- *A fixed theoretical vocabulary* (instrumentalism): D is the set of names in a formal theory, and we are simply reasoning about how we describe the world rather than about what exists. Under this reading, cK is a regimented language for expressing beliefs about the world, and $\forall x\,\Box Ex$ is a theorem of the language, not a claim about reality.

So, can one have cK without necessitism? *Yes, but only by reinterpreting the formal apparatus.* If cK is read with a natural-language reading—where "exists" in Ex means ordinary existence, and $\forall x$ ranges over actual things—then cK does commit one to necessitism. If one reinterprets terms or quantifiers (as Williamson, possibilists, or instrumentalists do), one can use the formal system without the metaphysical commitment.

Moral. The logic underdetermines the metaphysics. The same formal system can be given different interpretations, and what commits one to necessitism is not the theoremhood of $\forall x\,\Box Ex$ in cK but the interpretation of that theorem as a claim about actual existence.

Solution 5.3 (Exercise 32.3: Tense-logical counter-model for past-CBF2)**. Question.** Construct a variable-domain tense-logical model in which $H\exists x\,Px \to \exists x\,HPx$ fails.

Model. Let $W = \{t_1, t_2\}$, with $t_1 <_H t_2$ (reading $<_H$ as "t_1 is in the past of t_2"), so that from t_2 we can look back to t_1 using H ("it has been the case that"). Outer domain $D = \{\alpha\}$. Inner domains: $D_{t_1} = \{\alpha\}$ (Socrates existed in 450 BC), $D_{t_2} = \emptyset$ (no one we are naming exists now). $I(s) = \alpha$. $I(t_1, P) = \{\alpha\}$ (Socrates was a philosopher), $I(t_2, P) = \emptyset$.

Verification.

- $\mathcal{M}, t_2 \models H\exists x\,Px$: looking back to t_1, we have $\alpha \in D_{t_1}$ and $\alpha \in I(t_1, P)$, so $\exists x\,Px$ holds at t_1. Hence $H\exists x\,Px$ holds at t_2.
- $\mathcal{M}, t_2 \not\models \exists x\,HPx$: the inner domain at t_2 is empty, so no object serves as a witness to the existential *at* t_2.

Hence $H\exists x\,Px \to \exists x\,HPx$ fails at t_2.

Interpretation. It was once the case that someone was a philosopher (Socrates, back in 450 BC). But right now, there is no one of whom we can say that they were a philosopher (because, on the presentist picture modelled here, Socrates does not exist in the present to serve as a witness). This is the past analogue of the original Pryor-style failure of CBF2 in the modal case: the domain at the evaluation point may be too impoverished to contain a witness.

Remark. This model is the tense-logical sibling of the CBF2 counter-model in the modal case. The parallel underscores the claim made in Part VI (§25): eternalism and presentism stand to each other as necessitism and contingentism do, and the logic validates or refutes the Barcan-style formulas for exactly parallel reasons.

Solution 5.4 (Exercise 32.4: Constant-domain free logic?)**. Question.** Could one adopt a constant-domain semantics but still distinguish existing from non-existing objects by means of an existence predicate?

Discussion. Yes, and such systems have been studied. The idea: take the cK-style constant domain, but introduce a predicate E whose extension at each world $I(w, E)$ is a *proper subset* of D, varying from world to world. The domain itself is constant; the extension of "exists" is variable.

What the system looks like.

- Semantically, it is cK with a distinguished predicate E whose extension is not the full domain.
- The Barcan formulas are still valid (because the quantifiers range over the constant domain).
- But $\forall x \,\Box Ex$ is no longer valid: an object may be in the domain at every world without being in the *extension of* E at every world.
- Free-logic-style rules could be layered on top: quantifier instantiation restricted to E, existential generalization requiring E.

Philosophical motivation. Williamson's own view is essentially this. He takes the domain to contain all possibilia (abstract or concrete), and uses a predicate "is concrete" to carve out the subset of actually-located objects. The logic is cK (constant domain, Barcan formulas), but the distinction between "being in the domain" and "being concrete" does significant philosophical work. In this setup:

- $\forall x \,\Box$(is-in-the-domain) is trivially valid (everything is in the domain of every world, since the domain is constant).
- $\forall x \,\Box$(is-concrete) is invalid (I might have been non-concrete).

This captures the idea that necessitism is a thesis about being-in-the-domain, not about being-concrete, and it matches Williamson's preferred picture exactly.

Remark. This shows that the three axes—constant vs variable domains, Barcan vs no Barcan, existence-as-domain vs existence-as-predicate—are not as tightly coupled as one might naively think. The full space of options is larger and includes positions not usually discussed at an introductory level.

Solution 5.5 (Exercise 32.5: Is rigidity across non-existence defensible?)**. Question.** In vK, the Rigidity rule lets us derive $v : a = b$ from $w : a = b$ even when a's referent does not belong to the inner domain of v. Is this defensible?

Discussion. The rule requires us to accept that identity statements can hold at worlds where the referent does not exist—at least in the sense of not being in the inner domain. Three considerations bear on the question.

In favour.

(a) *Continuity of reference.* A proper name, once fixed to an object, should continue to refer to the same object even when we consider counterfactual scenarios where the object does not exist. When we say "Gandalf does not exist", we are still using "Gandalf" to refer to a specific individual (the wizard of Tolkien's fiction); otherwise the sentence would not have its intended meaning. If reference is continuous, identity statements involving the name are also continuous.

(b) *Formal elegance.* Rigidity makes the necessity of identity a theorem, which has considerable explanatory power: it accounts for Kripke's famous *a posteriori* necessities ("Hesperus = Phosphorus"), it respects the Leibniz law in modal contexts, and it connects smoothly with the theory of direct reference.

Against.

(a) *What is the truth-maker?* If a's referent does not exist in the inner domain of v, in what sense does $a = a$ hold at v? There is nothing there to be self-identical. One can reply: it is a purely logical truth, not a fact about what exists at v. But this reply presupposes a sharp distinction between logical and non-logical predicates, which is itself contestable.

(b) *Alternative systems are coherent.* Neutral free logic and contingent-identity systems are formally consistent and philosophically defensible. They give up either rigidity or the necessity of identity, but they do so for principled reasons—typically, to avoid the commitment that identity facts persist through non-existence.

Presupposition. The Rigidity rule, as applied in vK, presupposes that the identity of an object is not grounded in its presence at the world of evaluation. The identity of a is a cross-world invariant feature of a, not a feature that a acquires at each world where it exists. This is a substantive metaphysical thesis, sometimes called the *haecceitist* view of individual essences. Contingent-identity theorists reject it.

Verdict. Defensible, but not mandatory. The rule is philosophically loaded in vK in a way it is not in cK, and accepting it is accepting a particular metaphysics of reference and identity.

Solution 5.6 (Exercise 32.6: Sketching contingent-identity vK)**. Question.** Sketch a contingent-identity vK system. Which rules must be weakened? What happens to the necessity of identity?

Discussion. In a contingent-identity system, the Rigidity rule is abandoned. Two versions are salient.

Version 1: weak rigidity.
- Reflexivity: kept. $a = a$ holds at every world.
- Substitution: kept, but restricted to the current world.
- Rigidity: *abandoned*. From $w : a = b$, we cannot derive $v : a = b$.

Under this proposal, names can co-refer at one world and fail to co-refer at another. A classic case: Hesperus and Phosphorus (ancient astronomical names) are identical at our world, but in a world where the heavens look differently they might name different celestial bodies.

Consequences for $\forall x \forall y(x = y \to \Box(x = y))$: non-theorem. The proof in Part I used Rigidity essentially; without it, the proof fails. In fact, counter-models become available:
- Let $W = \{w_1, w_2\}$ with $w_1 R w_2$. $D = \{\alpha, \beta\}$. $I(w_1, a) = \alpha$, $I(w_1, b) = \alpha$ (so $a = b$ at w_1). But $I(w_2, a) = \alpha$, $I(w_2, b) = \beta$ (so $a \neq b$ at w_2).
- Here $a = b$ is true at w_1 but not at w_2, so $\Box(a = b)$ fails at w_1. The necessity of identity is therefore not a theorem.

Version 2: counterpart theory (Lewis). More radically, one can deny cross-world identity altogether. Instead of saying "the same individual exists at w_1 and w_2", one says "there is an individual at w_1 and a *counterpart* of it at w_2". Counterpart relations are world-specific and non-transitive. Under counterpart theory:
- There is no cross-world identity to speak of.
- "Necessarily Pa" means: at every world, a's counterpart there is P.
- The necessity of identity becomes false for a technical reason: identity requires a single object, counterpart relations do not deliver one.

What is lost and gained.
- *Lost.* The necessity of identity; the cleanest formal treatment of proper names; the simple account of *a posteriori* necessities (Hesperus/Phosphorus).
- *Gained.* Respect for the intuition that things could have been different—including, potentially, different identities. Flexibility in modelling world-relative identity phenomena (fission, fusion, ship-of-Theseus).

Verdict. Contingent-identity systems exist and are coherent. They give up a considerable amount of what makes Kripkean modal logic intuitive, and the trade-offs are substantial. Most contemporary philosophers of modality (Kripke, Williamson, Stalnaker, Soames) defend rigidity; the counterpart-theoretic alternative (Lewis) remains a serious option, though it has lost ground since the 1980s.