

Quantified Modal Logic with Constant Domains (cK)

Part I of Quantified Modal Logic

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Plan

What changes?

We extend modal propositional logic to a language with:

- individual variables $x, y, z, \dots$
- individual constants (proper names) $a, b, c, \dots$
- predicate symbols $P, Q, R, \dots$
- identity $=$
- the quantifiers $\forall, \exists$
- as before: $\neg, \wedge, \vee, \rightarrow, \Box, \Diamond$

New question

How do the quantifiers interact with the modal operators?

The central choice: constant vs. variable domains

In a Kripke model, what exactly do the quantifiers range over?

Constant domain (cK)

- One fixed domain D
- Same objects at every world
- Worlds differ only in *what is true of* the objects

Variable domain (vK)

- Each world has its own domain D_w
- Objects may exist at some worlds but not others
- Worlds differ in *what there is*

Today: cK.

Constant domain models

Definition

A **constant domain model** is a 4-tuple $\mathcal{M} = \langle W, R, D, I \rangle$:

- W : non-empty set of worlds
- $R \subseteq W \times W$: accessibility relation
- D : non-empty **constant** domain of discourse
- I : interpretation function
 - $I(c) \in D$ for each constant c (**same at every world**)
 - $I(w, R) \subseteq D^n$ for each n -ary predicate

Rigid designation

Constants receive the **same referent at every world**:

$$I(c) = \text{the same object in every } w \in W.$$

Names are called **rigid designators** (Kripke 1980).

Example

“Aristotle” refers to the same man at every possible world, even though what is true of him may differ.

Predicates, by contrast, have **world-relative extensions**:

$$I(w_1, P) \neq I(w_2, P) \text{ is possible.}$$

Truth conditions (1): atomic and propositional

Given an assignment $\sigma : \text{Vars} \rightarrow D$:

$$\mathcal{M}, w, \sigma \models R t_1 \dots t_n \iff \langle I_{w,\sigma}(t_1), \dots, I_{w,\sigma}(t_n) \rangle \in I(w, R)$$

$$\mathcal{M}, w, \sigma \models t_1 = t_2 \iff I_{w,\sigma}(t_1) = I_{w,\sigma}(t_2)$$

Boolean connectives: as in classical logic, at the same world.

- $\mathcal{M}, w, \sigma \models \neg\varphi$ iff $\mathcal{M}, w, \sigma \not\models \varphi$
- $\mathcal{M}, w, \sigma \models \varphi \wedge \psi$ iff both
- ...

Truth conditions (2): quantifiers and modals

Quantifiers range over the constant domain D :

$$\mathcal{M}, w, \sigma \models \forall x \varphi \quad \Longleftrightarrow \quad \text{for every } o \in D, \mathcal{M}, w, \sigma[x \mapsto o] \models \varphi$$

$$\mathcal{M}, w, \sigma \models \exists x \varphi \quad \Longleftrightarrow \quad \text{for some } o \in D, \mathcal{M}, w, \sigma[x \mapsto o] \models \varphi$$

Modal operators: as in propositional modal logic.

$$\mathcal{M}, w, \sigma \models \Box \varphi \quad \Longleftrightarrow \quad \text{for every } w' \text{ with } Rww', \mathcal{M}, w', \sigma \models \varphi$$

$$\mathcal{M}, w, \sigma \models \Diamond \varphi \quad \Longleftrightarrow \quad \text{for some } w' \text{ with } Rww', \mathcal{M}, w', \sigma \models \varphi$$

Assignment σ persists across worlds; the quantifier domain is the same everywhere.

De re vs. de dicto

Two readings of a modal claim:

- **De dicto:** modality attaches to a *proposition*.
 $\Box Pa$: “it is necessary *that* a is P ”.
- **De re:** modality attaches to an *object*.
 $\exists x \Box Px$: “there is something *such that* it is necessarily P ”.

Note

In cK, the open formula $\Box Px$, evaluated under σ with $\sigma(x) = o$, asks: does object o belong to $I(w', P)$ at every accessible w' ?

This is *de re*—a question about the object, not about a sentence.

cK as a formal system

$$\text{cK} = \text{K} + \text{classical first-order logic}$$

- **K** contributes the rules for $\Box$ and $\Diamond$ (from Part I of the course).
- **Classical FOL** contributes the rules for $\forall$, $\exists$, and $=$.
- The two sets of rules interact only through the **world-label discipline**: quantifier reasoning happens at a single world, just like propositional reasoning.

Two proof systems: **tableaux** and **labelled natural deduction**.

Tableau rules for quantifiers (cK)

Rule $T\forall$ (universal, no new name)

From $T : \forall x \varphi x, i$ and any constant c , add $T : \varphi c, i$. *May be re-applied.*

Rule $F\forall$ (existential, fresh name)

From $F : \forall x \varphi x, i$, introduce fresh c and add $F : \varphi c, i$.

Rule $T\exists$ (existential, fresh name)

From $T : \exists x \varphi x, i$, introduce fresh c and add $T : \varphi c, i$.

Rule $F\exists$ (universal, no new name)

From $F : \exists x \varphi x, i$ and any constant c , add $F : \varphi c, i$.

Order of rule application

- ➊ **Propositional rules** first—free, decompose existing formulas.
- ➋ **Existential rules** ($T\Diamond$, $F\Box$, $T\exists$, $F\forall$): each introduces fresh material (a world or a name).
Mark each such formula: *applied once only*.
- ➌ **Return to step 1**: fresh material may need decomposition.
- ➍ **Universal rules** ($T\Box$, $F\Diamond$, $T\forall$, $F\exists$):
 - $T\Box, i / F\Diamond, i$ at every j with iRj
 - $T\forall, i / F\exists, i$ with every constant on the branchDo **not** mark: may need re-applying.
- ➎ **Return to step 1**.

Principle: cheap things first, expensive things last, re-apply universals when new material appears.

Labelled ND rules for quantifiers (cK)

$\forall E$

From $w : \forall x \varphi x$, conclude $w : \varphi c$.

$\exists I$

From $w : \varphi c$, conclude $w : \exists x \varphi x$.

$\forall I$

If, for fresh c , we can derive $w : \varphi c$, conclude $w : \forall x \varphi x$.

$\exists E$

From $w : \exists x \varphi x$, assume $w : \varphi c$ (fresh c), derive χ not mentioning c , discharge.

Note: the world-label w is carried along but plays no role in the quantifier discipline itself—quantifier reasoning happens *at* a world.

Example: combining quantifier and modal rules

Claim: $\forall x \Box P_x \vdash_{cK} \Box Pa$.

1	$w : \forall x \Box P_x$	P
2	$w : \Box Pa$	$\forall E, 1$
3	$w R v$	fresh v
4	$v : Pa$	$\Box E, 2, 3$
5	$w : \Box Pa$	$\Box I, 3-4$

The proof shows the seamless **interleaving** of quantifier and modal rules.

Identity in a modal setting

Three tableau rules for $=$:

Reflexivity

At any index i , for any constant c : add $T : c = c, i$.

Substitution

From $T : a = b, i$ and any labelled formula involving a at i , substitute $a \leftrightarrow b$.

Rigidity

From $T : a = b, i$ and any index j : add $T : a = b, j$.

The **Rigidity rule** is what makes the modal case special: identity established at one world transfers to all worlds.

The necessity of identity

Theorem

$$\vdash_{cK} \forall x \forall y (x = y \rightarrow \Box(x = y))$$

Proof in labelled ND:

1		$w : a = b$	H
2		wRv	fresh v
3		$v : a = b$	Rigidity, 1
4		$w : \Box(a = b)$	$\Box I$, 2—3
5		$w : a = b \rightarrow \Box(a = b)$	$\rightarrow I$, 1—4
6		$w : \forall y (a = y \rightarrow \Box(a = y))$	$\forall I$, 5
7		$w : \forall x \forall y (x = y \rightarrow \Box(x = y))$	$\forall I$, 6

Barcan's formulas

Interactions between quantifiers and modal operators:

$$\mathbf{BF1} \quad \forall x \Box \varphi x \rightarrow \Box \forall x \varphi x$$

$$\mathbf{BF2} \quad \Diamond \exists x \varphi x \rightarrow \exists x \Diamond \varphi x$$

$$\mathbf{CBF1} \quad \Box \forall x \varphi x \rightarrow \forall x \Box \varphi x$$

$$\mathbf{CBF2} \quad \exists x \Diamond \varphi x \rightarrow \Diamond \exists x \varphi x$$

Theorem

All four are theorems of cK.

Due to **Ruth Barcan Marcus** (1946).

Proof of BF1: $\forall x \Box \varphi x \rightarrow \Box \forall x \varphi x$

1		$w : \forall x \Box \varphi x$	H
2		$w R v$	fresh v
3			arbitrary c
4		$w : \Box \varphi c$	$\forall E, 1$
5		$v : \varphi c$	$\Box E, 4, 2$
6		$v : \forall x \varphi x$	$\forall I, 3-5$
7		$w : \Box \forall x \varphi x$	$\Box I, 2-6$
8		$w : \forall x \Box \varphi x \rightarrow \Box \forall x \varphi x$	$\rightarrow I, 1-7$

Key: open the $\Box$ -subproof first (line 2), then pick an arbitrary c *inside* (line 3).

Proof of BF2: $\Diamond \exists x \varphi x \rightarrow \exists x \Diamond \varphi x$

1	$w : \Diamond \exists x \varphi x$	H
2	$w R v, v : \exists x \varphi x$	fresh v
3	$v : \varphi c$	fresh c
4	$w : \Diamond \varphi c$	$\Diamond I, 2, 3$
5	$w : \exists x \Diamond \varphi x$	$\exists I, 4$
6	$w : \exists x \Diamond \varphi x$	$\exists E, 2, 3-5$
7	$w : \exists x \Diamond \varphi x$	$\Diamond E, 1, 2-6$
8	$w : \Diamond \exists x \varphi x \rightarrow \exists x \Diamond \varphi x$	$\rightarrow I, 1-7$

Key: the fresh c from $\exists E$ at v can form $\Diamond\varphi c$ at w —because the domain is constant.

A non-theorem

Not a theorem of cK

$$\Diamond \exists x Px \rightarrow \Diamond \exists x (Px \wedge Qb)$$

The extra conjunct Qb —which does not involve the quantified variable—blocks the inference.

Counter-model: $W = \{w_1, w_2\}$, $w_1 R w_2$, $D = \{\alpha\}$, $I(b) = \alpha$, $I(w_2, P) = \{\alpha\}$, $I(w_2, Q) = \emptyset$.

- $\mathcal{M}, w_1 \models \Diamond \exists x Px$: at w_2 , α is P . ✓
- $\mathcal{M}, w_1 \not\models \Diamond \exists x (Px \wedge Qb)$: Qb fails at every accessible world. ✓

The existence predicate

Introduce a unary predicate E :

$$I(w, E) = D \quad \text{for every } w \in W.$$

Equivalently: $Ex =_{\text{Def}} \exists y(x = y)$.

Lemma

$$\models_{cK} \forall x (Ex \leftrightarrow \exists y(x = y))$$

At first sight the predicate E looks redundant in cK : its extension is the whole domain at every world.

But it lets us state a striking theorem...

Necessary existence in cK

Theorem

$$\models_{cK} \forall x \Box Ex$$

Whatever exists, necessarily exists.

1			arbitrary c
2		$w : c = c$	Reflexivity
3		wRv	fresh v
4		$v : c = c$	Rigidity, 2
5		$v : \exists y(c = y)$	$\exists I$, 4
6		$w : \Box \exists y(c = y)$	$\Box I$, 3—5
7		$w : \forall x \Box \exists y(x = y)$	$\forall I$, 1—6

The metaphysical face: necessitism

Necessitism

Necessarily, everything is necessarily something.

$$\Box \forall x \Box \exists y (x = y)$$

cK validates necessitism **by virtue of its semantics**:

- constant domain $\Rightarrow$ same objects at every world
- $\Rightarrow$ every object exists at every world
- $\Rightarrow$ existence is invariant under the modality

The Barcan formulas are the syntactic signature of this metaphysical commitment.

Summary

- $cK = K +$ classical FOL, with constant-domain semantics.
- Constants are **rigid designators**: same referent at every world.
- Predicates have **world-relative extensions**.
- Two proof systems carry over: tableaux and labelled natural deduction.
- **Barcan formulas** (BF1, BF2, CBF1, CBF2) are all theorems.
- Identity is necessary: $\forall x \forall y (x = y \rightarrow \Box(x = y))$.
- **Necessitism** is a theorem: $\forall x \Box Ex$.

Next time

What if we reject the constant-domain assumption?

Variable domains, free logic, and the failure of all four Barcan formulas.