Quantified Modal Logic with Constant Domains (cK)

Part I of Quantified Modal Logic

Pascal Ludwig

Sorbonne Université, SND (CNRS)

Plan

1. From Propositional to Quantified Modal Logic
2. Semantics for cK
3. cK as a Formal System
4. Identity
5. Barcan's Formulas
6. Existence and Necessity

Why modal logic? Applications in philosophy

Before extending to quantification, recall the breadth of what propositional modal logic already does:

- **Metaphysics of necessity**: $\Box, \Diamond$ for necessary/possible truth; different frame conditions = different conceptions of necessity (T, S4, S5).
- **Deontic logic**: $\Box\varphi$ as "it is obligatory that φ". Standard Deontic Logic = system D.
- **Epistemic logic** (Hintikka, 1962): $\Box\varphi$ as "the agent knows φ". Multi-agent extensions formalise common knowledge.
- **Provability logic** (Solovay, 1976): $\Box\varphi$ as "φ is provable in PA". Yields Gödel's second incompleteness theorem.
- **Tense logic** (Prior, 1957): G, F, H, P for past and future. Operators corresponding to different conceptions of time.

Why modal logic? Applications outside philosophy

The same apparatus reaches beyond philosophy:

- **Knowledge representation in AI**: description logics (foundation of OWL and the Semantic Web) are decidable fragments of modal logic.
- **Multi-agent systems and game theory**: epistemic and dynamic-epistemic logic for protocol analysis, coordination problems, common knowledge in equilibria.
- **Linguistics**: Kratzer-style semantics for modal verbs (*must*, *may*); Lewis-Stalnaker semantics for counterfactuals.
- **Cognitive modelling**: AGM belief revision; Pearl-style counterfactual causal reasoning.

Tense logic, in particular

Prior's 1957 *Time and Modality* → Pnueli's 1977 "Temporal Logic of Programs" → LTL/CTL model checking for chip and software verification (**Turing Award 2007**).

A direct line from philosophy of time to industrial-strength verification.

Why these applications work: decidability

Three concrete cases:

- **Intel** verifies a new processor design against a temporal-logic spec ("every memory request is eventually granted"). The model checker *always* returns: yes, or here is a counter-example.
- **SNOMED CT** (350,000 medical concepts) computes "myocardial infarction is a kind of cardiovascular disease" in milliseconds. Used in hospitals worldwide.
- **Cryptographic protocol verification**: an algorithm checks whether an attacker can ever learn the secret key. Halts with an answer.

What makes all this possible

Propositional modal logic is **decidable**: there exists an algorithm that, given any formula, terminates with the correct answer.

No guarantee of termination $\Rightarrow$ no industrial use. Decidability is the bridge from the seminar room to the chip factory.

What changes?

We extend modal propositional logic to a language with:

- individual variables $x, y, z, \ldots$
- individual constants (proper names) $a, b, c, \ldots$
- predicate symbols $P, Q, R, \ldots$
- identity $=$
- the quantifiers $\forall$, $\exists$
- as before: $\neg, \wedge, \vee, \rightarrow, \Box, \Diamond$

New question

How do the quantifiers interact with the modal operators?

The central choice: constant vs. variable domains

In a Kripke model, what exactly do the quantifiers range over?

Constant domain (cK)

- One fixed domain D
- Same objects at every world
- Worlds differ only in *what is true of* the objects

Variable domain (vK)

- Each world has its own domain D_w
- Objects may exist at some worlds but not others
- Worlds differ in *what there is*

Today: **cK**.

Constant domain models

Definition

A **constant domain model** is a 4-tuple $\mathcal{M} = \langle W, R, D, I \rangle$:

- W: non-empty set of worlds
- $R \subseteq W \times W$: accessibility relation
- D: non-empty **constant** domain of discourse
- I: interpretation function
  - $I(c) \in D$ for each constant c (**same at every world**)
  - $I(w, R) \subseteq D^n$ for each n-ary predicate

Rigid designation

Constants receive the **same referent at every world**:

$$I(c) = \text{the same object in every } w \in W.$$

Names are called **rigid designators** (Kripke 1980).

Example

"Aristotle" refers to the same man at every possible world, even though what is true of him may differ.

Predicates, by contrast, have **world-relative extensions**:

$$I(w_1, P) \neq I(w_2, P) \quad \text{is possible.}$$

Truth conditions (1): atomic and propositional

Given an assignment $\sigma : \text{Vars} \to D$:

$$\mathcal{M}, w, \sigma \models Rt_1 \dots t_n \iff \langle I_{w,\sigma}(t_1), \dots, I_{w,\sigma}(t_n) \rangle \in I(w, R)$$

$$\mathcal{M}, w, \sigma \models t_1 = t_2 \iff I_{w,\sigma}(t_1) = I_{w,\sigma}(t_2)$$

Boolean connectives: as in classical logic, at the same world.

- $\mathcal{M}, w, \sigma \models \neg\varphi$ iff $\mathcal{M}, w, \sigma \not\models \varphi$
- $\mathcal{M}, w, \sigma \models \varphi \wedge \psi$ iff both
- ...

Truth conditions (2): quantifiers and modals

Quantifiers range over the constant domain D:

$$\mathcal{M}, w, \sigma \models \forall x\, \varphi \quad \Longleftrightarrow \quad \text{for every } o \in D,\ \mathcal{M}, w, \sigma[x \mapsto o] \models \varphi$$

$$\mathcal{M}, w, \sigma \models \exists x\, \varphi \quad \Longleftrightarrow \quad \text{for some } o \in D,\ \mathcal{M}, w, \sigma[x \mapsto o] \models \varphi$$

Modal operators: as in propositional modal logic.

$$\mathcal{M}, w, \sigma \models \Box\varphi \quad \Longleftrightarrow \quad \text{for every } w' \text{ with } Rww',\ \mathcal{M}, w', \sigma \models \varphi$$

$$\mathcal{M}, w, \sigma \models \Diamond\varphi \quad \Longleftrightarrow \quad \text{for some } w' \text{ with } Rww',\ \mathcal{M}, w', \sigma \models \varphi$$

Assignment σ persists across worlds; the quantifier domain is the same everywhere.

De re vs. de dicto

Two readings of a modal claim:

- **De dicto**: modality attaches to a *proposition*.
  $\Box Pa$: "it is necessary *that a* is P".
- **De re**: modality attaches to an *object*.
  $\exists x \Box Px$: "there is something *such that* it is necessarily P".

Note

In cK, the open formula $\Box Px$, evaluated under σ with $\sigma(x) = o$, asks: does object o belong to $I(w', P)$ at every accessible w'?

This is *de re*—a question about the object, not about a sentence.

cK as a formal system

$$\boxed{\mathsf{cK} = \mathsf{K} + \text{classical first-order logic}}$$

- **K** contributes the rules for $\Box$ and $\Diamond$ (from Part I of the course).
- **Classical FOL** contributes the rules for $\forall$, $\exists$, and $=$.
- The two sets of rules interact only through the **world-label discipline**: quantifier reasoning happens at a single world, just like propositional reasoning.

Two proof systems: **tableaux** and **labelled natural deduction**.

Tableau rules for quantifiers (cK)

Rule T$\forall$ (universal, no new name)

From T : $\forall x\,\varphi x,\ i$ and any constant c, add T : $\varphi c,\ i$. *May be re-applied.*

Rule F$\forall$ (existential, fresh name)

From F : $\forall x\,\varphi x,\ i$, introduce fresh c and add F : $\varphi c,\ i$.

Rule T$\exists$ (existential, fresh name)

From T : $\exists x\,\varphi x,\ i$, introduce fresh c and add T : $\varphi c,\ i$.

Rule F$\exists$ (universal, no new name)

From F : $\exists x\,\varphi x,\ i$ and any constant c, add F : $\varphi c,\ i$.

Order of rule application

1. **Propositional rules** first—free, decompose existing formulas.
2. **Existential rules** (T$\Diamond$, F$\Box$, T$\exists$, F$\forall$): each introduces fresh material (a world or a name).
   Mark each such formula: *applied once only*.
3. **Return to step 1**: fresh material may need decomposition.
4. **Universal rules** (T$\Box$, F$\Diamond$, T$\forall$, F$\exists$):
   - T$\Box$, i / F$\Diamond$, i at every j with iRj
   - T$\forall$, i / F$\exists$, i with every constant on the branch

   Do **not** mark: may need re-applying.
5. **Return to step 1**.

Principle: cheap things first, expensive things last, re-apply universals when new material appears.

Labelled ND rules for quantifiers (cK)

$\forall$E

From $w : \forall x\, \varphi x$, conclude $w : \varphi c$.

$\exists$I

From $w : \varphi c$, conclude $w : \exists x\, \varphi x$.

$\forall$I

If, for fresh c, we can derive $w : \varphi c$, conclude $w : \forall x\, \varphi x$.

$\exists$E

From $w : \exists x\, \varphi x$, assume $w : \varphi c$ (fresh c), derive χ not mentioning c, discharge.

Note: the world-label w is carried along but plays no role in the quantifier discipline itself—quantifier reasoning happens *at* a world.

Example: combining quantifier and modal rules

Claim: $\forall x \Box Px \vdash_{cK} \Box Pa$.

1	$w : \forall x \Box Px$	P
2	$w : \Box Pa$	$\forall$E, 1
3	$\quad wRv$	fresh v
4	$\quad v : Pa$	$\Box$E, 2, 3
5	$w : \Box Pa$	$\Box$I, 3—4

The proof shows the seamless **interleaving** of quantifier and modal rules.

Identity in a modal setting

Three tableau rules for $=$:

Reflexivity

At any index i, for any constant c: add $\mathsf{T} : c = c,\ i$.

Substitution

From $\mathsf{T} : a = b,\ i$ and any labelled formula involving a at i, substitute $a \leftrightarrow b$.

Rigidity

From $\mathsf{T} : a = b,\ i$ and any index j: add $\mathsf{T} : a = b,\ j$.

The Rigidity rule is what makes the modal case special: identity established at one world transfers to all worlds.

The necessity of identity

Theorem

$\vdash_{cK} \forall x \forall y (x = y \to \Box(x = y))$

Proof in labelled ND:

1	$w : a = b$	H
2	$\quad wRv$	fresh v
3	$\quad v : a = b$	Rigidity, 1
4	$w : \Box(a = b)$	$\Box$I, 2—3
5	$w : a = b \to \Box(a = b)$	$\to$I, 1—4
6	$w : \forall y(a = y \to \Box(a = y))$	$\forall$I, 5
7	$w : \forall x \forall y(x = y \to \Box(x = y))$	$\forall$I, 6

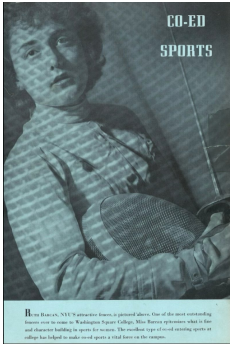
Barcan's formulas

Interactions between quantifiers and modal operators:

BF1	$\forall x \Box \varphi x \rightarrow \Box \forall x \, \varphi x$
BF2	$\Diamond \exists x \, \varphi x \rightarrow \exists x \Diamond \varphi x$
CBF1	$\Box \forall x \, \varphi x \rightarrow \forall x \Box \varphi x$
CBF2	$\exists x \Diamond \varphi x \rightarrow \Diamond \exists x \, \varphi x$

Theorem

All four are theorems of cK.

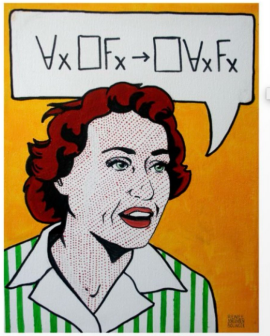


Ruth Barcan as an NYU undergraduate fencer, c. 1941. She formulated the four formulas in 1946, while still a doctoral student.

Proof of BF1: $\forall x \Box \varphi x \to \Box \forall x \, \varphi x$

1	$w : \forall x \Box \varphi x$	H
2	wRv	fresh v
3		arbitrary c
4	$w : \Box \varphi c$	$\forall$E, 1
5	$v : \varphi c$	$\Box$E, 4, 2
6	$v : \forall x \, \varphi x$	$\forall$I, 3—5
7	$w : \Box \forall x \, \varphi x$	$\Box$I, 2—6
8	$w : \forall x \Box \varphi x \to \Box \forall x \, \varphi x$	$\to$I, 1—7

Key: open the $\Box$-subproof first (line 2), then pick an arbitrary c *inside* (line 3).



Pop-art portrait of Ruth Barcan Marcus, with BF1 (artist: Renée Bolinger).

Proof of BF2: $\Diamond\exists x\,\varphi x \to \exists x\,\Diamond\varphi x$

1	$w : \Diamond\exists x\,\varphi x$	H
2	$wRv,\ v : \exists x\,\varphi x$	fresh v
3	$v : \varphi c$	fresh c
4	$w : \Diamond\varphi c$	$\Diamond$I, 2, 3
5	$w : \exists x\,\Diamond\varphi x$	$\exists$I, 4
6	$w : \exists x\,\Diamond\varphi x$	$\exists$E, 2, 3—5
7	$w : \exists x\,\Diamond\varphi x$	$\Diamond$E, 1, 2—6
8	$w : \Diamond\exists x\,\varphi x \to \exists x\,\Diamond\varphi x$	$\to$I, 1—7

Key: the fresh c from $\exists$E at v can form $\Diamond\varphi c$ at w—because the domain is constant.

A non-theorem

Not a theorem of cK

$$\Diamond\exists x\, Px \rightarrow \Diamond\exists x\, (Px \wedge Qb)$$

The extra conjunct Qb—which does not involve the quantified variable—blocks the inference.

Counter-model: $W = \{w_1, w_2\}$, $w_1 R w_2$, $D = \{\alpha\}$, $I(b) = \alpha$, $I(w_2, P) = \{\alpha\}$, $I(w_2, Q) = \emptyset$.

- $\mathcal{M}, w_1 \models \Diamond\exists x\, Px$: at w_2, α is P. ✓
- $\mathcal{M}, w_1 \not\models \Diamond\exists x(Px \wedge Qb)$: Qb fails at every accessible world. ✓

The existence predicate

Introduce a unary predicate E:

$$I(w, E) = D \quad \text{for every } w \in W.$$

Equivalently: $Ex =_{\mathrm{Def}} \exists y(x = y)$.

Lemma

$\models_{cK} \forall x(Ex \leftrightarrow \exists y(x = y))$

At first sight the predicate E looks redundant in cK: its extension is the whole domain at every world.

But it lets us state a striking theorem...

Necessary existence in cK

Theorem

$\models_{cK} \forall x \Box Ex$

Whatever exists, necessarily exists.

1	| |	arbitrary c
2	| | $w : c = c$	Reflexivity
3	| | wRv	fresh v
4	| | $v : c = c$	Rigidity, 2
5	| | $v : \exists y(c = y)$	$\exists$I, 4
6	| $w : \Box\exists y(c = y)$	$\Box$I, 3—5
7	$w : \forall x \Box\exists y(x = y)$	$\forall$I, 1—6

The metaphysical face: necessitism

Necessitism

Necessarily, everything is necessarily something.

$$\Box\forall x\,\Box\exists y(x = y)$$

cK validates necessitism **by virtue of its semantics**:

- constant domain $\Rightarrow$ same objects at every world
- $\Rightarrow$ every object exists at every world
- $\Rightarrow$ existence is invariant under the modality

The Barcan formulas are the syntactic signature of this metaphysical commitment.

Ruth Barcan Marcus (1921–2012).

Summary

- **cK** = K + classical FOL, with constant-domain semantics.
- Constants are **rigid designators**: same referent at every world.
- Predicates have **world-relative extensions**.
- Two proof systems carry over: tableaux and labelled natural deduction.
- **Barcan formulas** (BF1, BF2, CBF1, CBF2) are all theorems.
- Identity is necessary: $\forall x \forall y (x = y \rightarrow \Box(x = y))$.
- **Necessitism** is a theorem: $\forall x \, \Box E x$.

Next time

What if we reject the constant-domain assumption?

Variable domains, free logic, and the failure of all four Barcan formulas.