Modal Correspondence Exercises: Solution

Exercise 1: Partial functionality and $\Diamond p \to \Box p$

Theorem 1. *For every frame $\mathcal{F}$,*

$$\mathcal{F} \models \Diamond p \to \Box p \quad \iff \quad \mathcal{F} \textit{ is partially functional.}$$

Partial functionality: $\forall w \forall u \forall v\, ((Rwu \wedge Rwv) \to u = v)$. *Every world has* at most one *successor.*

($\Leftarrow$): Partial functionality implies validity of $\Diamond p \to \Box p$

Suppose $\mathcal{F}$ is partially functional. Let $\mathcal{M}$ be any model on $\mathcal{F}$, $w \in W$. Suppose $\mathcal{M}, w \models \Diamond p$. We want $\mathcal{M}, w \models \Box p$.

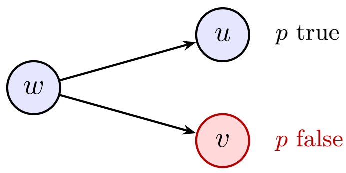
Since $\mathcal{M}, w \models \Diamond p$: there exists u with Rwu and $\mathcal{M}, u \models p$.

Now let v be any world with Rwv. By partial functionality, Rwu and Rwv imply $u = v$. Therefore $\mathcal{M}, v \models p$.

Since v was arbitrary, $\mathcal{M}, w \models \Box p$. $\square$

($\Rightarrow$): Validity of $\Diamond p \to \Box p$ implies partial functionality

Contrapositive: suppose $\mathcal{F}$ is not partially functional. Then $\exists w, u, v$: Rwu, Rwv, $u \neq v$.



Define $I(p, u) = T$ and $I(p, v) = F$. (Well-defined since $u \neq v$.)

Then $\mathcal{M}, w \models \Diamond p$ (via u) but $\mathcal{M}, w \not\models \Box p$ (because of v). So $\mathcal{M}, w \not\models \Diamond p \to \Box p$. $\square$

Exercise 2: Functionality and $\Diamond p \leftrightarrow \Box p$

Theorem 2. *For every frame $\mathcal{F}$,*

$$\mathcal{F} \models \Diamond p \leftrightarrow \Box p \quad \Longleftrightarrow \quad \mathcal{F} \text{ is functional.}$$

Functionality: $\forall w \, \exists v \, \forall u \, (Rwu \leftrightarrow u = v)$. *Every world has* exactly one *successor.*

Note that $\Diamond p \leftrightarrow \Box p$ is $(\Diamond p \to \Box p) \wedge (\Box p \to \Diamond p)$, and functionality is partial functionality + seriality. The first conjunct corresponds to partial functionality (Exercise 1); the second to seriality (axiom D). So the result follows by combining the two. We verify this directly.

($\Leftarrow$): Functionality implies validity of $\Diamond p \leftrightarrow \Box p$

Suppose $\mathcal{F}$ is functional. Then every w has exactly one successor v_0: Rwu iff $u = v_0$.

($\to$): Suppose $\mathcal{M}, w \models \Diamond p$. Then $\mathcal{M}, v_0 \models p$. For any u with Rwu: $u = v_0$, so $\mathcal{M}, u \models p$. Hence $\mathcal{M}, w \models \Box p$.

($\Leftarrow$): Suppose $\mathcal{M}, w \models \Box p$. Since Rwv_0: $\mathcal{M}, v_0 \models p$. So $\mathcal{M}, w \models \Diamond p$.

Therefore $\mathcal{M}, w \models \Diamond p \leftrightarrow \Box p$. □

($\Rightarrow$): Validity of $\Diamond p \leftrightarrow \Box p$ implies functionality

Contrapositive: if $\mathcal{F}$ is not functional, then either (a) some w has no successor, or (b) some w has two distinct successors.

Case (a): w has no successor. Then $\mathcal{M}, w \models \Box p$ (vacuously) but $\mathcal{M}, w \not\models \Diamond p$ (no successor). So $\Box p \to \Diamond p$ fails.

Case (b): w has two distinct successors $u \neq v$. Set $I(p, u) = T$, $I(p, v) = F$. Then $\mathcal{M}, w \models \Diamond p$ but $\mathcal{M}, w \not\models \Box p$. So $\Diamond p \to \Box p$ fails.

In both cases, $\mathcal{F} \not\models \Diamond p \leftrightarrow \Box p$. □

Exercise 3: Density and $\Box\Box p \to \Box p$

Theorem 3. *For every frame $\mathcal{F}$,*

$$\mathcal{F} \models \Box\Box p \to \Box p \quad \Longleftrightarrow \quad \mathcal{F} \textit{ is dense.}$$

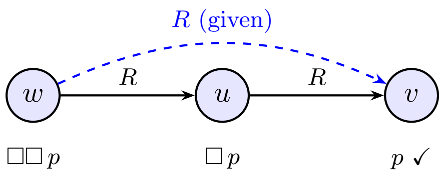
Density: $\forall u \forall v\ (Ruv \to \exists w\ (Ruw \land Rwv))$. Between any two related worlds, there is an intermediate world.

Remark 1. Compare with axiom 4 ($\Box p \to \Box\Box p$), which corresponds to *transitivity*. The two conditions are "converses": transitivity says two-step paths have a shortcut; density says one-step paths have a detour.

($\Leftarrow$): Density implies validity of $\Box\Box p \to \Box p$

Suppose $\mathcal{F}$ is dense. Let $\mathcal{M}$ be any model on $\mathcal{F}$, $w \in W$. Suppose $\mathcal{M}, w \models \Box\Box p$. We want $\mathcal{M}, w \models \Box p$.

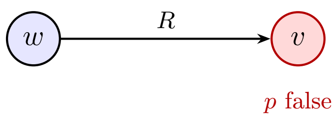
Let v be any world with Rwv. By density, there exists u with Rwu and Ruv.



Since $\mathcal{M}, w \models \Box\Box p$ and Rwu: $\mathcal{M}, u \models \Box p$.
Since $\mathcal{M}, u \models \Box p$ and Ruv: $\mathcal{M}, v \models p$.
Since v was arbitrary, $\mathcal{M}, w \models \Box p$. $\Box$

($\Rightarrow$): Validity of $\Box\Box p \to \Box p$ implies density

Contrapositive: suppose $\mathcal{F}$ is not dense. Then $\exists w, v$: Rwv but no intermediate world (no u with Rwu and Ruv).



Define $I(p, v) = F$ and $I(p, x) = T$ for all $x \neq v$.

Claim: $\mathcal{M}, w \models \Box\Box p$.

Let u be any successor of w. We need every successor t of u to satisfy p. Could $t = v$? That would require Rwu and Ruv, making u an intermediate world between w and v. But no such world exists. So no successor-of-a-successor of w is v, and every such world satisfies p. Therefore $\mathcal{M}, w \models \Box\Box p$.

But Rwv and $\mathcal{M}, v \not\models p$, so $\mathcal{M}, w \not\models \Box p$.

Therefore $\mathcal{M}, w \not\models \Box\Box p \to \Box p$. $\Box$

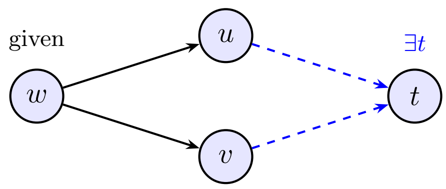
Exercise 4: Confluence and $\Diamond\Box p \to \Box\Diamond p$

Theorem 4. *For every frame $\mathcal{F}$,*

$$\mathcal{F} \models \Diamond\Box p \to \Box\Diamond p \quad\iff\quad \mathcal{F} \text{ is confluent.}$$

Confluence: $\forall w \forall u \forall v\ ((Rwu \wedge Rwv) \to \exists t\ (Rut \wedge Rvt))$. Any two successors of a world have a common successor.

This is also known as the *Church–Rosser* or *diamond property.*

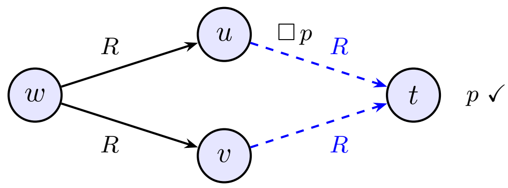


$(\Leftarrow)$: Confluence implies validity of $\Diamond\Box p \to \Box\Diamond p$

Suppose $\mathcal{F}$ is confluent. Let $\mathcal{M}$ be any model on $\mathcal{F}$, $w \in W$. Suppose $\mathcal{M}, w \models \Diamond\Box p$. We want $\mathcal{M}, w \models \Box\Diamond p$.

Since $\mathcal{M}, w \models \Diamond\Box p$: $\exists u$ with Rwu and $\mathcal{M}, u \models \Box p$.

Let v be any world with Rwv. We need $\mathcal{M}, v \models \Diamond p$.



By confluence, since Rwu and Rwv: $\exists t$ with Rut and Rvt.

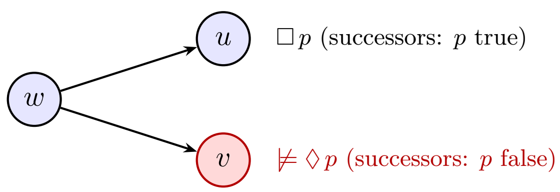
Since $\mathcal{M}, u \models \Box p$ and Rut: $\mathcal{M}, t \models p$.

Since Rvt and $\mathcal{M}, t \models p$: $\mathcal{M}, v \models \Diamond p$.

Since v was arbitrary, $\mathcal{M}, w \models \Box\Diamond p$. $\square$

$(\Rightarrow)$: Validity of $\Diamond\Box p \to \Box\Diamond p$ implies confluence

Contrapositive: suppose $\mathcal{F}$ is not confluent. Then $\exists w, u, v$: Rwu, Rwv, but no common successor (no t with Rut and Rvt).



Define I such that:

- for every t with Rut: $I(p, t) = T$;
- for every t with Rvt: $I(p, t) = F$.

This is well-defined because u and v have no common successor: no t needs both values.

Then:

- $\mathcal{M}, u \models \Box p$: every successor of u has p true.
- $\mathcal{M}, w \models \Diamond\Box p$: via u.
- $\mathcal{M}, v \not\models \Diamond p$: every successor of v has p false (and if v has no successors, $\Diamond p$ fails trivially).

- $\mathcal{M}, w \not\models \Box\Diamond p$: because of v.

Therefore $\mathcal{M}, w \not\models \Diamond\Box p \to \Box\Diamond p$. $\square$

Remark 2. The well-definedness of the valuation is the crux of the ($\Rightarrow$) direction. The absence of a common successor is exactly what lets us assign contradictory truth-values to the successor sets of u and v. This is a general pattern: the structural defect of R is what makes the counter-model constructible.