Modal Correspondence Theory

Plan for today

1. **Background**: Frames, models, frame properties
2. **The central idea**: What is a "correspondence"?
3. **Caution!** Model validity $\neq$ frame validity
4. **The five correspondences**: D, M, 4, B, 5
   - Model-theoretic proofs (both directions)
   - Proof-theoretic derivations (one direction only!)
5. **Applications**: Deontic, tense, and epistemic logic
6. **Circular time**: A concrete limit of modal definability
7. **Upshot**: Why modal logic?

Frame

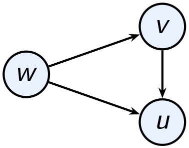
$\mathcal{F} = \langle W, R \rangle$

- W: non-empty set of worlds
- $R \subseteq W \times W$: accessibility

Model

$\mathcal{M} = \langle W, R, I \rangle$

- I: valuation of atoms at each world

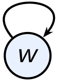


Key semantic clauses:

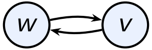
- $\mathcal{M}, w \models \Box\varphi$ iff φ holds at *every* accessible world
- $\mathcal{M}, w \models \Diamond\varphi$ iff φ holds at *some* accessible world

Properties of Frames

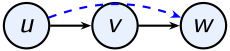
Property	Condition on R	Intuition
Reflexive	$\forall w\ Rww$	every world sees itself
Transitive	$Ruv \wedge Rvw \Rightarrow Ruw$	chains compose
Symmetric	$Rvw \Rightarrow Rwv$	access is bidirectional
Euclidean	$Ruv \wedge Ruw \Rightarrow Rvw$	siblings see each other
Serial	$\forall w\, \exists v\ Rwv$	no dead ends



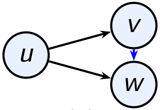
Reflexive



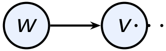
Symmetric



Transitive



Euclidean



Serial

Modal Correspondence — The Definition

Definition

A modal formula φ **corresponds to** a frame property P iff:

$$\text{for every frame } \mathcal{F}, \quad \mathcal{F} \models \varphi \iff \mathcal{F} \text{ has property } P.$$

Unpack $\mathcal{F} \models \varphi$:

$$\mathcal{F} \models \varphi \quad \overset{\text{def}}{\iff} \quad \underbrace{\text{for \textbf{every} model } \mathcal{M} \text{ on } \mathcal{F}}_{\text{every valuation } I}, \underbrace{\text{for \textbf{every} world } w}_{\text{every index}}, \ \mathcal{M}, w \models \varphi.$$

Two universal quantifiers! The formula must hold

- at every world,
- under every possible interpretation of the atoms.

Caution!

What the definition does NOT say

If a *particular model* $\mathcal{M}$ satisfies $\Box\varphi \to \varphi$, then its frame is reflexive.

This is FALSE!

What the definition DOES say

If *every possible model* on a frame $\mathcal{F}$ satisfies the formula at every world, then $\mathcal{F}$ is reflexive.

The difference: one lucky interpretation vs. every conceivable interpretation.

Counter-example: a model satisfies M, but the frame isn't reflexive

Consider the simplest possible frame:

$$W = \{w\}, \qquad R = \emptyset.$$

w $\qquad \neg Rww$ — **not reflexive!**

Set $I(w, p) = T$ for every variable p. Check $\mathcal{M}, w \models \Box p \to p$:

1. $\mathcal{M}, w \models \Box p$? $\quad R = \emptyset$: no accessible worlds.
   Universal quantifier over $\emptyset \Rightarrow$ **vacuously true**. $\quad \checkmark$
2. $\mathcal{M}, w \models p$? $\quad$ Yes: $I(w, p) = T$. $\quad \checkmark$
3. So $\mathcal{M}, w \models \Box p \to p$. $\quad$ (True $\to$ True.)

The model satisfies axiom M at every world. Yet the frame is not reflexive!

Why the *frame* doesn't validate M

Same frame ($W = \{w\}$, $R = \emptyset$), **different valuation**: $I'(w, p) = F$.

Check $\mathcal{M}', w \models \Box p \to p$:

1. $\mathcal{M}', w \models \Box p$: still vacuously true. $\checkmark$
2. $\mathcal{M}', w \models p$: **No!** $I'(w, p) = F$. $\times$
3. So $\mathcal{M}', w \not\models \Box p \to p$. (True $\to$ False $=$ False.)

The moral

The first model was a **lucky interpretation**: p happened to be true, making the conditional trivially satisfied. The second model **exposed the real problem**: $\Box p$ can be vacuously true while p is false, because w cannot see itself.

Rule of Thumb

When working with correspondence theory, always ask:

- Does the formula hold because of the **structure of R**?
  $\longrightarrow$ frame property.
- Or does it hold because of a **particular choice of I**?
  $\longrightarrow$ model accident — says nothing about the frame.

This is why the ($\Rightarrow$) direction of each correspondence proof works by **contrapositive**: if R lacks the structural property, we explicitly *construct a valuation I* that falsifies the formula.

Overview

Axiom	Formula	Frame property
D	$\Box\varphi \to \Diamond\varphi$	Seriality
M (T)	$\Box\varphi \to \varphi$	Reflexivity
4	$\Box\varphi \to \Box\Box\varphi$	Transitivity
B	$\varphi \to \Box\Diamond\varphi$	Symmetry
5	$\Diamond\varphi \to \Box\Diamond\varphi$	Euclideanness

For each, we prove two directions:

($\Leftarrow$) Property $\Rightarrow$ formula valid. (by *reductio*)

($\Rightarrow$) Formula valid $\Rightarrow$ property. (by *contrapositive*: construct counter-model)

Then a proof-theoretic derivation using labelled ND.

D Defines Serial Frames

Theorem

$\mathcal{F} \models \Box\varphi \to \Diamond\varphi \iff \mathcal{F}$ is serial.

($\Leftarrow$) Seriality $\Rightarrow$ validity of D. By reductio.

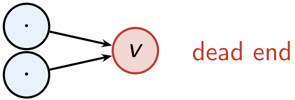
Suppose $\mathcal{M}, w \not\models \Box\varphi \to \Diamond\varphi$. Then:

- $\mathcal{M}, w \models \Box\varphi$ and $\mathcal{M}, w \not\models \Diamond\varphi$.

Seriality: $\exists w'$ with Rww'. By (1): $\mathcal{M}, w' \models \varphi$. By (2): $\mathcal{M}, w' \models \neg\varphi$. Contradiction. $\Box$

($\Rightarrow$) Validity of $D \Rightarrow$ seriality. Contrapositive.

If $\neg$serial: $\exists v$ with no successor (dead end).



Then $\mathcal{M}, v \models \Box\varphi$ (vacuously) and $\mathcal{M}, v \not\models \Diamond\varphi$. So $\mathcal{F} \not\models D$. $\Box$

D: Proof-theoretic derivation

Add to K a **seriality rule**: assert wRv for fresh v.

1	$w : \Box \varphi$	H
2	wRv (fresh v, by seriality)	
3	$v : \varphi$	$\Box$E, 1, 2
4	$w : \Diamond \varphi$	$\Diamond$I, 2, 3
5	$w : \Box \varphi \to \Diamond \varphi$	$\to$I, 1—4

Key step: line 2. In pure K, we cannot assert a successor exists. The seriality rule is what makes the derivation go through.

Important remark: the ND proof gives ($\Leftarrow$) only!

($\Leftarrow$): proof-theoretic

If R is serial, the axiom is **derivable**.

ND shows: structural rule **sufficient** to derive the axiom.

($\Rightarrow$): model-theoretic only

If the axiom is valid, then R **must** be serial.

This **cannot** be done proof-theoretically!

Why? The ($\Rightarrow$) direction says: validity of a formula *forces* a structural property on R.

Its proof works by contrapositive: assume R lacks the property, then construct a specific valuation I that falsifies the formula. Choosing a valuation is a *semantic* operation — a proof system manipulates formulas, it doesn't reason about which valuations exist.

This asymmetry (syntax $\leftrightarrow$ semantics) holds for every correspondence result.

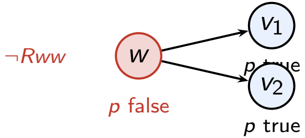
M (T) Defines Reflexive Frames

Theorem

$\mathcal{F} \models \Box\varphi \to \varphi \iff \mathcal{F}$ is reflexive.

($\Leftarrow$) Suppose $\mathcal{F}$ reflexive, $\mathcal{M}, w \models \Box\varphi$. Since Rww: $\mathcal{M}, w \models \varphi$. $\Box$

($\Rightarrow$) Contrapositive: $\exists w, \neg Rww$. Define $I(p, w) = F$; $I(p, i) = T$ for all Rwi.



Then $\mathcal{M}, w \models \Box p$ (every successor of w has p true) but $\mathcal{M}, w \not\models p$. $\Box$

M: Proof-theoretic derivation

Add to K a **reflexivity rule**: at any world w, assert wRw.

1	$w : \Box\varphi$	H
2	wRw	Refl.
3	$w : \varphi$	$\Box$E, 1, 2
4	$w : \Box\varphi \to \varphi$	$\to$I, 1—3

Key step: line 2. Reflexivity gives wRw, which feeds into standard $\Box$E.

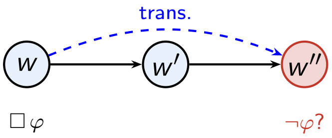
In K: line 2 is unavailable. We have $\Box\varphi$ at w but no link from w to itself, so we can't extract φ at w.

Axiom 4 Defines Transitive Frames ($S4$)

Theorem

$\mathcal{F} \models \Box\varphi \to \Box\Box\varphi \iff \mathcal{F}$ is transitive.

($\Leftarrow$) By reductio. Suppose $\mathcal{M}, w \models \Box\varphi$ but $\mathcal{M}, w \models \Diamond\Diamond\neg\varphi$.
Then $\exists w', w''$: Rww', $Rw'w''$, $\mathcal{M}, w'' \models \neg\varphi$.
Transitivity: Rww''. But $\mathcal{M}, w \models \Box\varphi$ gives $\mathcal{M}, w'' \models \varphi$. Contradiction. $\Box$



($\Rightarrow$) Contrapositive: $\exists u, v, w$: Ruv, Rvw, $\neg Ruw$.
Set $I(p, w) = F$, all other successors of u have p true.
Then $\mathcal{M}, u \models \Box p$ but $\mathcal{M}, u \models \Diamond\Diamond\neg p$. $\Box$

$S4$: Proof-theoretic derivation

$S4 = T+$ **transitivity rule**: from wRv and vRu, conclude wRu.

1	$w : \Box p$	P
2	wRv	fresh v
3	vRu	fresh u
4	wRu	Trans., 2, 3
5	$u : p$	$\Box$E, 1, 4
6	$v : \Box p$	$\Box$I, 3—5
7	$w : \Box\Box p$	$\Box$I, 2—6

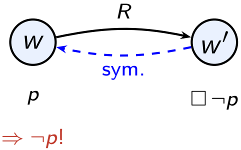
Key step: line 4. Transitivity gives wRu from $wRv + vRu$, which lets $\Box$E bring p from w down to u. Without this: stuck.

Axiom B Defines Symmetric Frames

Theorem

$\mathcal{F} \models \varphi \to \Box\Diamond\varphi \iff \mathcal{F}$ is symmetric.

($\Leftarrow$) By reductio. Suppose $\mathcal{M}, w \models p$ but $\mathcal{M}, w \not\models \Box\Diamond p$.
Then $\mathcal{M}, w \models \Diamond\Box\neg p$: $\exists w'$, Rww', $\mathcal{M}, w' \models \Box\neg p$.
Symmetry: $Rw'w$. So $\mathcal{M}, w \models \neg p$. Contradiction. $\Box$



($\Rightarrow$) Contrapositive: $\exists v, w$: Rvw, $\neg Rwv$. Set $I(p, v) = T$, successors of w have p false. Then $\mathcal{M}, v \models p$ but $\mathcal{M}, v \not\models \Box\Diamond p$. $\Box$

B: Proof-theoretic derivation

$B = T +$ **symmetry rule**: from wRv, conclude vRw.

1	$w : p$	P
2	wRv	fresh v
3	vRw	Sym., 2
4	$v : \Diamond p$	$\Diamond$I, 3, 1
5	$w : \Box\Diamond p$	$\Box$I, 2—4

Key step: line 3. Symmetry reverses the arrow: vRw. Then $\Diamond$I uses $vRw + w : p$ to get $v : \Diamond p$.

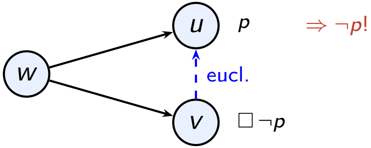
Note: at line 4, we use $w : p$ (premise, label w) inside the $\Box$I sub-proof. This is **not** an illegal import: we don't relabel it at v, we use it *at* w via the reversed link vRw.

Axiom 5 Defines Euclidean Frames ($S5$)

Theorem

$\mathcal{F} \models \Diamond\varphi \to \Box\Diamond\varphi \iff \mathcal{F}$ is euclidean.

($\Leftarrow$) By reductio. Suppose $\mathcal{M}, w \models \Diamond p$ and $\mathcal{M}, w \models \Diamond\Box\neg p$.
Then $\exists u$: Rwu, $u \models p$; and $\exists v$: Rwv, $v \models \Box\neg p$.
Euclidean: $Rwu \wedge Rwv \Rightarrow Rvu$. So $u \models \neg p$. Contradiction. $\Box$



($\Rightarrow$) Contrapositive: $\exists u, v, w$: Ruv, Ruw, $\neg Rvw$. Set $I(p, w) = T$, successors of v false. Then $u \models \Diamond p$ but $u \not\models \Box\Diamond p$. $\Box$

$S5$: Proof-theoretic derivation

$S5 = T+$ **euclideanness rule**: from wRv and wRu, conclude vRu.

1	$w : \Diamond p$	P
2	$wRv, \quad v : p$	fresh v
3	wRu	fresh u
4	uRv	Eucl., 3, 2
5	$u : \Diamond p$	$\Diamond$I, 4, 2
6	$w : \Box\Diamond p$	$\Box$I, 3—5
7	$w : \Box\Diamond p$	$\Diamond$E, 1, 2—6

Nested structure: $\Box$I (lines 3–5) inside $\Diamond$E (lines 2–6).

Key step: line 4. Euclideanness gives uRv. Then $\Diamond$I: u sees v (the p-world), so $u : \Diamond p$.

$S5$ and Equivalence Relations

$S5 = T +$ axiom 5

Accessibility is reflexive + euclidean.

Reflexive + euclidean $\Rightarrow$ also symmetric and transitive.
So R is an **equivalence relation**: worlds fall into *clusters*, fully connected within each cluster.

Collapse of modalities in $S5$

- $\Box\Box\varphi \leftrightarrow \Box\varphi$ and $\Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$
- $\Box\Diamond\varphi \leftrightarrow \Diamond\varphi$ and $\Diamond\Box\varphi \leftrightarrow \Box\varphi$
- Every string of $\Box$'s and $\Diamond$'s reduces to a single $\Box$ or $\Diamond$.

Summary

System	Axiom	Frame condition	ND rule
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	none	—
D	$\Box\varphi \to \Diamond\varphi$	seriality	Serial. rule
T	$\Box\varphi \to \varphi$	reflexivity	Refl. rule
$S4$	$\Box\varphi \to \Box\Box\varphi$	refl. + trans.	Refl. + Trans.
B	$\varphi \to \Box\Diamond\varphi$	refl. + sym.	Refl. + Sym.
$S5$	$\Diamond\varphi \to \Box\Diamond\varphi$	equivalence	Refl. + Eucl.

Model-theoretic proofs

Both directions ($\Leftarrow$ and $\Rightarrow$).
$\Rightarrow$ by contrapositive: construct a bad I.

Proof-theoretic derivations

($\Leftarrow$) direction only.
Structural rule $\Rightarrow$ axiom derivable.

Tense Logic — The Philosophical Question

Question

The standard temporal ordering ("occurring before") is a *dense, strict linear order without endpoints*.

Can tense logic axiomatize all of these properties?

Two pairs of operators:

- **Future**: $G\varphi$ ("always going to be") / $F\varphi$ ("will be")
- **Past**: $H\varphi$ ("has always been") / $P\varphi$ ("has been")

R_F = "strictly later than", $\quad R_P$ = converse of R_F.

Interaction axioms (past = converse of future):

$$\varphi \to GP\varphi \qquad \varphi \to HF\varphi$$

The Target: Properties of Temporal Order

What should a temporal ordering satisfy? (Russell, Walker, Prior)

Property	**Condition**	**Intuition**
Transitivity	$a < b \wedge b < c \Rightarrow a < c$	order chains compose
Irreflexivity	$\neg(a < a)$	no instant before itself
Asymmetry	$a < b \Rightarrow \neg(b < a)$	no temporal circles
Seriality	no first, no last instant	time extends both ways
Connectivity	$a \neq b \Rightarrow a < b \vee b < a$	time is a line
Density	$a < b \Rightarrow \exists c\,(a < c < b)$	always an instant between

Together: **dense strict linear order without endpoints**.
Standard model: $(\mathbb{Q}, <)$.

Which of these six properties can tense logic define?

What Tense Logic Can Capture

Property	Tense axiom	Why it works
Transitivity	$G\varphi \to GG\varphi$	axiom 4 for G
No last instant	$G\varphi \to F\varphi$	axiom D for G
No first instant	$H\varphi \to P\varphi$	axiom D for H
Connectivity	$(PF\varphi \vee FP\varphi) \to$	zig-zag $\Rightarrow$
	$(P\varphi \vee \varphi \vee F\varphi)$	past, present, or future
Density	$Fp \to FFp$	intermediate instant

Density explained

"If p will be the case, then it will be that it will be the case" — there must be an intermediate instant between now and the p-instant.

Taking Stock: What a Propositional Language Can Do

The tense-logical axiom system:

K axioms for G and H	(basic tense logic)
$\varphi \to GP\varphi, \quad \varphi \to HF\varphi$	(past = converse of future)
$G\varphi \to GG\varphi, \quad H\varphi \to HH\varphi$	(transitivity)
$G\varphi \to F\varphi, \quad H\varphi \to P\varphi$	(no endpoints)
$(PF\varphi \vee FP\varphi) \to (P\varphi \vee \varphi \vee F\varphi)$	(linearity)
$Fp \to FFp, \quad Pp \to PPp$	(density)

characterizes exactly the class of **dense linear orders without endpoints**.

A remarkable fact

A purely *propositional* language — just two pairs of modal operators, no quantifiers, no individual variables — captures the topology of $(\mathbb{Q}, <)$.

But two properties on our checklist remain...

What Tense Logic Cannot Capture

Two properties remain: **irreflexivity** and **asymmetry**.
These are precisely what makes the order *strict*: "before" vs. "before or simultaneous."

Theorem (Goldblatt–Thomason)

Irreflexivity ($\forall w\, \neg Rww$) **cannot be defined by any modal formula**.

Similarly for asymmetry ($Rwv \to \neg Rvw$).

Why? Modal formulas are *bisimulation-invariant*.
Irreflexivity is not preserved by bisimulations.

More precisely: a frame class is modally definable only if it is closed under *ultrafilter extensions*.
The class of irreflexive frames is not.

The Philosophical Moral

What modal logic captures

The **topology** of time:

- dense
- linear
- no endpoints
- transitive

What it misses

The **strictness** of precedence:

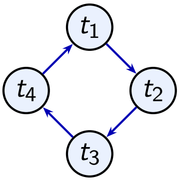
- irreflexivity
- asymmetry

Tense logic cannot distinguish $<$ from $\leq$.

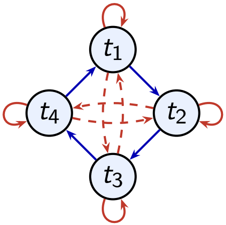
A dense linear preorder without endpoints validates *exactly the same formulas* as $(\mathbb{Q}, <)$.

The very property that makes temporal ordering "temporal" —
that no instant precedes itself —
is precisely what escapes the expressive power of modal logic.

Circular Time: A Concrete Consequence



(a) The cycle



(b) Transitive closure

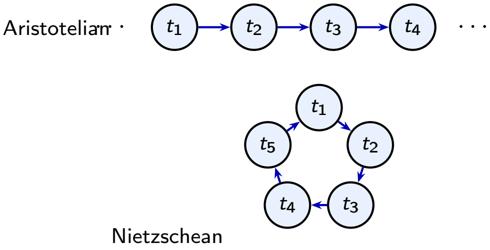
Since asymmetry is not modally definable, nothing prevents time from looping.

Key observation. Transitivity ($G\varphi \to GG\varphi$) forces the cycle to collapse:

$$t_1 R\, t_2,\ t_2 R\, t_3 \;\Rightarrow\; t_1 R\, t_3 \;\Rightarrow\; \cdots \;\Rightarrow\; t_1 R\, t_1 \quad \text{(self-loop!)}$$

The transitive closure is a **complete graph**: every instant sees every instant, including itself — a single $S5$ **equivalence class**.

Aristotelian Time vs. Nietzschean Time



Both validate **exactly the same** tense-logical formulas (for a large enough cycle).

The difference is real, but **modally invisible**.

Why? Modal logic is *local*: a formula of depth k sees k steps ahead. On a cycle of size $n > k$, every instant has the same local view as a point in $\mathbb{Z}$.

Upshot: Why Modal Logic?

Modal logic is not a poor approximation to first-order logic.

- **Mathematical identity**
  Bisimulation-invariant fragment of FOL (van Benthem, 1976).
- **Decidability**
  K, T, $S4$, $S5$ are decidable. FOL is not (Church, 1936).
- **Perspectival reasoning**
  Captures reasoning *from a standpoint*; the anonymity of worlds is a feature.
- **Informative limitations**
  Correspondence theory exists *because* of the restriction. The missing row (irreflexivity) is a discovery, not a defect.

What modal logic *cannot* say is often as philosophically revealing as what it can.