Modal Logic — Exercise Sheet 2

Solutions

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1 Exercise 1: Natural Deduction in K

1.1 (a) $\vdash_K \Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$

Left-to-right: $\Box(p \wedge q) \vdash_K \Box p \wedge \Box q$**.**

1	$w : \Box(p \wedge q)$	P
2	$\quad \mid\ wRv$	fresh v
3	$\quad \mid\ v : p \wedge q$	$\Box$E, 1, 2
4	$\quad \mid\ v : p$	$\wedge$E, 3
5	$w : \Box p$	$\Box$I, 2—4
6	$\quad \mid\ wRu$	fresh u
7	$\quad \mid\ u : p \wedge q$	$\Box$E, 1, 6
8	$\quad \mid\ u : q$	$\wedge$E, 7
9	$w : \Box q$	$\Box$I, 6—8
10	$w : \Box p \wedge \Box q$	$\wedge$I, 5, 9

Right-to-left: $\Box p \wedge \Box q \vdash_K \Box(p \wedge q)$**.**

1	$w : \Box p \wedge \Box q$	P
2	$w : \Box p$	$\wedge$E, 1
3	$w : \Box q$	$\wedge$E, 1
4	$\quad \mid\ wRv$	fresh v
5	$\quad \mid\ v : p$	$\Box$E, 2, 4
6	$\quad \mid\ v : q$	$\Box$E, 3, 4
7	$\quad \mid\ v : p \wedge q$	$\wedge$I, 5, 6
8	$w : \Box(p \wedge q)$	$\Box$I, 4—7

1.2 (b) $\vdash_K (\Box p \lor \Box q) \to \Box(p \lor q)$

1	$\mid\ \mid\ \underline{w : \Box p \lor \Box q}$	H
2	$\mid\ \mid\ \mid\ \underline{w : \Box p}$	H ($\lor$E)
3	$\mid\ \mid\ \mid\ \mid\ \underline{wRv}$	fresh v
4	$\mid\ \mid\ \mid\ \mid\ v : p$	$\Box$E, 2, 3
5	$\mid\ \mid\ \mid\ \mid\ v : p \lor q$	$\lor$I, 4
6	$\mid\ \mid\ \mid\ w : \Box(p \lor q)$	$\Box$I, 3—5
7	$\mid\ \mid\ \mid\ \underline{w : \Box q}$	H ($\lor$E)
8	$\mid\ \mid\ \mid\ \mid\ \underline{wRu}$	fresh u
9	$\mid\ \mid\ \mid\ \mid\ u : q$	$\Box$E, 7, 8
10	$\mid\ \mid\ \mid\ \mid\ u : p \lor q$	$\lor$I, 9
11	$\mid\ \mid\ \mid\ w : \Box(p \lor q)$	$\Box$I, 8—10
12	$\mid\ \mid\ w : \Box(p \lor q)$	$\lor$E, 1, 2—6, 7—11
13	$\mid\ w : (\Box p \lor \Box q) \to \Box(p \lor q)$	$\to$I, 1—12

Each case of $\lor$E opens its own $\Box$I sub-proof, bringing the relevant variable down to the accessible world.

1.3 (c) $\vdash_K (\Diamond p \lor \Diamond q) \leftrightarrow \Diamond(p \lor q)$

Left-to-right: $\Diamond p \lor \Diamond q \vdash_K \Diamond(p \lor q)$.

1	$\mid\ \underline{w : \Diamond p \lor \Diamond q}$	P
2	$\mid\ \mid\ \underline{w : \Diamond p}$	H ($\lor$E)
3	$\mid\ \mid\ \mid\ \underline{wRv,\ v : p}$	fresh v
4	$\mid\ \mid\ \mid\ v : p \lor q$	$\lor$I, 3
5	$\mid\ \mid\ \mid\ w : \Diamond(p \lor q)$	$\Diamond$I, 3, 4
6	$\mid\ \mid\ w : \Diamond(p \lor q)$	$\Diamond$E, 2, 3—5
7	$\mid\ \mid\ \underline{w : \Diamond q}$	H ($\lor$E)
8	$\mid\ \mid\ \mid\ \underline{wRu,\ u : q}$	fresh u
9	$\mid\ \mid\ \mid\ u : p \lor q$	$\lor$I, 8
10	$\mid\ \mid\ \mid\ w : \Diamond(p \lor q)$	$\Diamond$I, 8, 9
11	$\mid\ \mid\ w : \Diamond(p \lor q)$	$\Diamond$E, 7, 8—10
12	$\mid\ w : \Diamond(p \lor q)$	$\lor$E, 1, 2—6, 7—11

Right-to-left: $\Diamond(p \vee q) \vdash_K \Diamond p \vee \Diamond q$.

1	$w : \Diamond(p \vee q)$	P
2	$\quad wRv,\ v : p \vee q$	fresh v
3	$\quad\quad v : p$	H ($\vee$E)
4	$\quad\quad w : \Diamond p$	$\Diamond$I, 2, 3
5	$\quad\quad w : \Diamond p \vee \Diamond q$	$\vee$I, 4
6	$\quad\quad v : q$	H ($\vee$E)
7	$\quad\quad w : \Diamond q$	$\Diamond$I, 2, 6
8	$\quad\quad w : \Diamond p \vee \Diamond q$	$\vee$I, 7
9	$\quad w : \Diamond p \vee \Diamond q$	$\vee$E, 2, 3—5, 6—8
10	$w : \Diamond p \vee \Diamond q$	$\Diamond$E, 1, 2—9

1.4 (d) $\vdash_K \Diamond(p \wedge q) \to (\Diamond p \wedge \Diamond q)$

This was proved in the lecture notes (Section 5.5). We reproduce it:

1	$w : \Diamond(p \wedge q)$	P
2	$\quad wRv,\ v : p \wedge q$	fresh v
3	$\quad v : p$	$\wedge$E, 2
4	$\quad v : q$	$\wedge$E, 2
5	$\quad w : \Diamond p$	$\Diamond$I, 2, 3
6	$\quad w : \Diamond q$	$\Diamond$I, 2, 4
7	$\quad w : \Diamond p \wedge \Diamond q$	$\wedge$I, 5, 6
8	$w : \Diamond p \wedge \Diamond q$	$\Diamond$E, 1, 2—7

1.5 (e) $\vdash_K \Box\neg\Box\neg\neg p \to \Box\neg\Box p$

1	$\mid\ \mid\ w : \Box\neg\Box\neg\neg p$	H
2	$\mid\ \mid\ \mid\ wRv$	fresh v
3	$\mid\ \mid\ \mid\ v : \neg\Box\neg\neg p$	$\Box$E, 1, 2
4	$\mid\ \mid\ \mid\ \mid\ v : \Box p$	H ($\neg$I)
5	$\mid\ \mid\ \mid\ \mid\ \mid\ vRu$	fresh u
6	$\mid\ \mid\ \mid\ \mid\ \mid\ u : p$	$\Box$E, 4, 5
7	$\mid\ \mid\ \mid\ \mid\ \mid\ \mid\ u : \neg p$	H ($\neg$I)
8	$\mid\ \mid\ \mid\ \mid\ \mid\ \mid\ u : \bot$	$\neg$E, 6, 7
9	$\mid\ \mid\ \mid\ \mid\ \mid\ u : \neg\neg p$	$\neg$I, 7—8
10	$\mid\ \mid\ \mid\ \mid\ v : \Box\neg\neg p$	$\Box$I, 5—9
11	$\mid\ \mid\ \mid\ \mid\ v : \bot$	$\neg$E, 3, 10
12	$\mid\ \mid\ \mid\ v : \neg\Box p$	$\neg$I, 4—11
13	$\mid\ \mid\ w : \Box\neg\Box p$	$\Box$I, 2—12
14	$\mid\ w : \Box\neg\Box\neg\neg p \to \Box\neg\Box p$	$\to$I, 1—13

Key idea: Assume $\Box p$ at v; inside a nested $\Box$I sub-proof, derive $\neg\neg p$ at each accessible world u (by double negation introduction from p). This gives $\Box\neg\neg p$ at v, contradicting $\neg\Box\neg\neg p$. So $\neg\Box p$ at v.

1.6 (f) Derived rule: from $\neg\Box\varphi$, derive $\Diamond\neg\varphi$

We prove $\neg\Box\varphi \vdash_K \Diamond\neg\varphi$:

1	$\mid\ w : \neg\Box\varphi$	P
2	$\mid\ \mid\ w : \neg\Diamond\neg\varphi$	H ($\neg$I)
3	$\mid\ \mid\ \mid\ wRv$	fresh v
4	$\mid\ \mid\ \mid\ \mid\ v : \neg\varphi$	H ($\neg$I)
5	$\mid\ \mid\ \mid\ \mid\ w : \Diamond\neg\varphi$	$\Diamond$I, 3, 4
6	$\mid\ \mid\ \mid\ \mid\ w : \bot$	$\neg$E, 2, 5
7	$\mid\ \mid\ \mid\ v : \neg\neg\varphi$	$\neg$I, 4—6
8	$\mid\ \mid\ \mid\ v : \varphi$	$\neg\neg$E, 7
9	$\mid\ \mid\ w : \Box\varphi$	$\Box$I, 3—8
10	$\mid\ \mid\ w : \bot$	$\neg$E, 1, 9
11	$\mid\ w : \neg\neg\Diamond\neg\varphi$	$\neg$I, 2—10
12	$\mid\ w : \Diamond\neg\varphi$	$\neg\neg$E, 11

Strategy: Assume $\neg\Diamond\neg\varphi$ for reductio. Inside a $\Box$I sub-proof, derive φ at each accessible world (by assuming $\neg\varphi$, deriving $\Diamond\neg\varphi$, contradiction). This gives $\Box\varphi$, contradicting the premise.

2 Exercise 2: Natural Deduction in Alethic Logics

2.1 2(a): Theorems of M $(= T)$

$\vdash_M \varphi \to \Diamond\varphi$.

1	$\mid\ \mid\ \underline{w : \varphi}$	H
2	$\mid\ \mid\ \mid\ \underline{w : \neg\Diamond\varphi}$	H ($\neg$I)
3	$\mid\ \mid\ \mid\ w : \Box\neg\varphi$	Def. $\Diamond$, 2
4	$\mid\ \mid\ \mid\ w : \neg\varphi$	$\Box$E-M, 3
5	$\mid\ \mid\ \mid\ w : \bot$	$\neg$E, 1, 4
6	$\mid\ \mid\ w : \neg\neg\Diamond\varphi$	$\neg$I, 2—5
7	$\mid\ \mid\ w : \Diamond\varphi$	$\neg\neg$E, 6
8	$\mid\ w : \varphi \to \Diamond\varphi$	$\to$I, 1—7

Line 3 uses the equivalence $\neg\Diamond\varphi \dashv\vdash \Box\neg\varphi$ (proved in Ex. 1f and lecture notes). Line 4 uses $\Box$E-M (the reflexivity rule).

$\vdash_M \Box\varphi \to \Diamond\varphi$.

1	$\mid\ \mid\ \underline{w : \Box\varphi}$	H
2	$\mid\ \mid\ w : \varphi$	$\Box$E-M, 1
3	$\mid\ \mid\ w : \Diamond\varphi$	$\varphi \to \Diamond\varphi$, 2
4	$\mid\ w : \Box\varphi \to \Diamond\varphi$	$\to$I, 1—3

Line 3 applies the theorem just proved. Alternatively: from $\Box\varphi$, $\Box$E-M gives φ; then repeat the reductio proof above.

2.2 2(b): Theorems of $S4$, $S5$, and B

In $S4$: $\vdash_{S4} \Box\Box\varphi \leftrightarrow \Box\varphi$. ($\Rightarrow$): $\Box\Box\varphi \to \Box\varphi$ by $\Box$E-M (from T, which $S4$ extends).

($\Leftarrow$): $\Box\varphi \to \Box\Box\varphi$ (axiom 4). Proved in the lecture notes using transitivity:

1	$\mid\ \underline{w : \Box\varphi}$	P
2	$\mid\ \mid\ \underline{wRv}$	fresh v
3	$\mid\ \mid\ \mid\ \underline{vRu}$	fresh u
4	$\mid\ \mid\ \mid\ wRu$	Trans., 2, 3
5	$\mid\ \mid\ \mid\ u : \varphi$	$\Box$E, 1, 4
6	$\mid\ \mid\ v : \Box\varphi$	$\Box$I, 3—5
7	$\mid\ w : \Box\Box\varphi$	$\Box$I, 2—6

In $S4$: $\vdash_{S4} \Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$. ($\Rightarrow$): $\Diamond\Diamond\varphi \to \Diamond\varphi$.

1	$w : \Diamond\Diamond\varphi$	H
2	$\quad wRv,\ v : \Diamond\varphi$	fresh v
3	$\quad\quad vRu,\ u : \varphi$	fresh u
4	$\quad\quad wRu$	Trans., 2, 3
5	$\quad\quad w : \Diamond\varphi$	$\Diamond$I, 4, 3
6	$\quad w : \Diamond\varphi$	$\Diamond$E, 2, 3—5
7	$w : \Diamond\varphi$	$\Diamond$E, 1, 2—6
8	$w : \Diamond\Diamond\varphi \to \Diamond\varphi$	$\to$I, 1—7

($\Leftarrow$): $\Diamond\varphi \to \Diamond\Diamond\varphi$.

1	$w : \Diamond\varphi$	H
2	$\quad wRv,\ v : \varphi$	fresh v
3	$\quad vRv$	Refl.
4	$\quad v : \Diamond\varphi$	$\Diamond$I, 3, 2
5	$\quad w : \Diamond\Diamond\varphi$	$\Diamond$I, 2, 4
6	$w : \Diamond\Diamond\varphi$	$\Diamond$E, 1, 2—5
7	$w : \Diamond\varphi \to \Diamond\Diamond\varphi$	$\to$I, 1—6

In $S4$: $\vdash_{S4} \Box\Diamond\Diamond\varphi \leftrightarrow \Box\Diamond\varphi$. This follows from $\Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$ (just proved) by "lifting under $\Box$": since the equivalence is a theorem, necessitation gives $\Box(\Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi)$, and distribution gives $\Box\Diamond\Diamond\varphi \leftrightarrow \Box\Diamond\varphi$.

More explicitly: the left-to-right direction $\Box\Diamond\Diamond\varphi \to \Box\Diamond\varphi$ is proved by opening a $\Box$I subproof at fresh v, applying $\Box$E to get $\Diamond\Diamond\varphi$ at v, then applying $\Diamond\Diamond\varphi \to \Diamond\varphi$ at v to get $\Diamond\varphi$ at v. Similarly for the converse.

In $S5$: $\vdash_{S5} \Box\Diamond\varphi \leftrightarrow \Diamond\varphi$. ($\Rightarrow$): $\Box\Diamond\varphi \to \Diamond\varphi$ by $\Box$E-M.

($\Leftarrow$): $\Diamond\varphi \to \Box\Diamond\varphi$ is axiom 5. $\Box$

In $S5$: prove axiom B ($\varphi \to \Box\Diamond\varphi$).

1	$w : \varphi$	H
2	$w : \Diamond\varphi$	$\varphi \to \Diamond\varphi$, 1
3	$w : \Box\Diamond\varphi$	Axiom 5, 2
4	$w : \varphi \to \Box\Diamond\varphi$	$\to$I, 1—3

Line 2: $\varphi \to \Diamond\varphi$ is a theorem of M (proved in 2a), and $S5$ extends M. Line 3: axiom 5 ($\Diamond\psi \to \Box\Diamond\psi$).

In B: $\vdash_B \Diamond\Box\varphi \to \varphi$. *Hint: use $\neg\varphi \to \Box\Diamond\neg\varphi$ (axiom B with $\neg\varphi$) and exercise 1e.*

We prove the contrapositive $\neg\varphi \to \neg\Diamond\Box\varphi$, i.e. $\neg\varphi \to \Box\neg\Box\varphi$:

1. From $\neg\varphi$, by axiom B: $\Box\Diamond\neg\varphi$, i.e. $\Box\neg\Box\neg\neg\varphi$ (using $\Diamond\psi = \neg\Box\neg\psi$ with $\psi = \neg\varphi$, so $\Diamond\neg\varphi = \neg\Box\neg\neg\varphi$).
2. By exercise 1e: $\Box\neg\Box\neg\neg\varphi \to \Box\neg\Box\varphi$.
3. Chain: $\neg\varphi \to \Box\neg\Box\varphi$.

Since $\Box\neg\Box\varphi$ is $\neg\Diamond\Box\varphi$ (by the equivalence $\neg\Diamond\psi \dashv\vdash \Box\neg\psi$ with $\psi = \Box\varphi$), we obtain $\neg\varphi \to \neg\Diamond\Box\varphi$. Contrapositive: $\Diamond\Box\varphi \to \varphi$. $\square$

In $S5$: $\vdash_{S5} \Diamond\Box\varphi \leftrightarrow \Box\varphi$. *Hint: use exercise 2a, exercise 1f, and axiom 5 in the form $\Diamond\neg\varphi \to \Box\Diamond\neg\varphi$.*

($\Leftarrow$): $\Box\varphi \to \Diamond\Box\varphi$ by $\psi \to \Diamond\psi$ (theorem of M, with $\psi = \Box\varphi$).

($\Rightarrow$): $\Diamond\Box\varphi \to \Box\varphi$. We prove the contrapositive $\neg\Box\varphi \to \neg\Diamond\Box\varphi$:

1. From $\neg\Box\varphi$, by exercise 1f: $\Diamond\neg\varphi$.
2. By axiom 5 (with $\neg\varphi$): $\Diamond\neg\varphi \to \Box\Diamond\neg\varphi$.
3. So $\Box\Diamond\neg\varphi$, i.e. $\Box\neg\Box\neg\neg\varphi$ (by definition of $\Diamond$).
4. By exercise 1e: $\Box\neg\Box\neg\neg\varphi \to \Box\neg\Box\varphi$.
5. So $\Box\neg\Box\varphi$, which is $\neg\Diamond\Box\varphi$.

Contrapositive: $\Diamond\Box\varphi \to \Box\varphi$. $\square$

3 Exercise 3: Deontic Logic and Tense Logic

3.1 3(a): In $O + OO$, prove $Op \to OPp$

The axiom OO is $Op \leftrightarrow OOp$. Axiom D for O is $O\psi \to P\psi$ (i.e. $O\psi \to \neg O\neg\psi$).

1	$\quad\mid\ w : Op$	H
2	$\quad\mid\ w : OOp$	OO ($\Rightarrow$), 1
3	$\quad\mid\quad\mid\ wRv$	fresh v
4	$\quad\mid\quad\mid\ v : Op$	OE, 2, 3
5	$\quad\mid\quad\mid\ v : Pp$	D, 4
6	$\quad\mid\ w : OPp$	OI, 3—5
7	$w : Op \to OPp$	$\to$I, 1—6

Explanation: From Op, axiom OO gives OOp (line 2). Open an O-introduction sub-proof at fresh v: O-elimination on OOp gives Op at v (line 4); axiom D at v gives Pp (line 5). Close: OPp (line 6).

3.2 3(b): Theorems of tense logic K_t

In tense logic, G ("always future") and H ("always past") are $\Box$-operators with respect to the future and past accessibility relations R_F and R_P, which are **converses**: wR_Fv iff vR_Pw.

Converse rule. From wR_Fv, derive vR_Pw (and vice versa).

$\vdash_{K_t} PGp \to p$.

1	$w : PGp$	H
2	$wR_Pv,\ v : Gp$	fresh v (PE)
3	vR_Fw	Converse, 2
4	$w : p$	GE, 2, 3
5	$w : p$	PE, 1, 2—4
6	$w : PGp \to p$	→I, 1—5

Explanation: PGp at w means there is a past world v where Gp holds. Since v is in the past of w, w is in the future of v (converse, line 3). Since Gp holds at v and w is in v's future, G-elimination gives p at w (line 4).

$\vdash_{K_t} FHp \to p$.

1	$w : FHp$	H
2	$wR_Fv,\ v : Hp$	fresh v (FE)
3	vR_Pw	Converse, 2
4	$w : p$	HE, 2, 3
5	$w : p$	FE, 1, 2—4
6	$w : FHp \to p$	→I, 1—5

Completely symmetric: FHp at w means there is a future world v where Hp holds. Since w is in the past of v (converse), H-elimination gives p at w.