Modal logic — exercise sheet 2

Exercice 1 : Natural deduction in K
Prove the following theorems (from no premise):

a) $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$
b) $(\Box p \vee \Box q) \rightarrow \Box(p \vee q)$
c) $(\Diamond p \vee \Diamond q) \leftrightarrow \Diamond(p \vee q)$
d) $\Diamond(p \wedge q) \rightarrow \Diamond p \wedge \Diamond q$
e) $\Box\neg\Box\neg\neg p \rightarrow \Box\neg\Box p$
f) Prove that the following rule is valide in K : if $\neg\Box\phi$ occurs in a proof at line *n*, one can write $\Diamond\neg\phi$ at line *m*.

Exercice 2 : Natural deduction in alethic logics
a) Prove the following theorems:
- $\phi \rightarrow \Diamond\phi$ in M
- $\Box\phi \rightarrow \Diamond\phi$ in M
- $\Box\phi \rightarrow \phi$ in K+$\phi \rightarrow \Diamond\phi$

b) Prove the following theorems :
- $\Box\Box\phi \leftrightarrow \Box\phi$ in S4
- $\Diamond\Diamond\phi \leftrightarrow \Diamond\phi$ in S4
- $\Box\Diamond\Diamond\phi \leftrightarrow \Box\Diamond\phi$ in S4
- $\Box\Diamond\phi \leftrightarrow \Diamond\phi$ in S5
- Prove axiom B ($\phi \rightarrow \Box\Diamond\phi$) In S5
- $\Diamond\Box\phi \rightarrow \phi$ in B (*Hint* : use : $\neg\phi \rightarrow \Box\Diamond\neg\phi$, as well as exercise 1.e).
- $\Diamond\Box\phi \leftrightarrow \Box\phi$ in S (cf. Ex 2a. and Ex 1f, then use axiom 5 in this form : $\Diamond\neg\phi \rightarrow \Box\Diamond\neg\phi$)
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Exercice 3 : Natural deduction in deontic logic (O) and tense logic (Kt)
a) Let use consider the following new axiom for deontic logic: (($\mathbf{OO}$) : $\mathbf{O}p \leftrightarrow \mathbf{OO}p$. In ($\mathbf{O} + \mathbf{OO}$), prove that : $\mathbf{O}p \rightarrow \mathbf{OP}p$
b) Prove the following theorems of Kt :
- $\mathbf{PG}p \rightarrow p$
- $\mathbf{FH}p \rightarrow p$