Modal Logic II

A Natural Deduction System for K and Extensions

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Outline

From Tableaux to Proofs

In Modal Logic I, we studied the **semantics** of K:

- Kripke models
- The tableau method (validity testing, counter-models)

We now turn to **proof theory**: a formal system in which valid arguments can be *derived* step by step.

Approach: *labelled natural deduction*.

- Every formula carries a **world label**
- **Accessibility assertions** wRv are handled explicitly
- Directly connected to the Kripke semantics and tableaux we already know

Labelled Formulae

Labelled formula

An expression $w : \varphi$, where w is a **world label** and φ a modal formula.
Read: "φ holds at world w."

Accessibility statement

An expression wRv, where w, v are world labels.
Read: "v is accessible from w."

A proof consists of a sequence of labelled formulae and accessibility statements, each justified by a rule.

Propositional Rules

All standard propositional ND rules ($\wedge$, $\vee$, $\to$, $\neg$, $\bot$) carry over unchanged, **provided all formulae share the same world label**.

Propositional reasoning operates *within a single world*.

Constraint

One cannot mix labels. For instance, $w : \varphi$ and $v : \varphi \to \psi$ do *not* give $v : \psi$: the modus ponens must happen at a single world.

Exception: Negation Introduction

The "no mixing" constraint has one important exception: **negation introduction** ($\neg$I) allows labels to be mixed.

If assuming $v : \varphi$ leads to $\perp$ at *any* world, the assumption is refuted:

1	$\quad\quad v : \varphi$	H
2	$\quad\quad \vdots$	
3	$\quad\quad w : \perp$	$(w \neq v)$
4	$v : \neg\varphi$	$\neg$I, 1—3

Why?

$\perp$ is not world-relative. If any world in a model harbours a contradiction, the entire model is impossible—so the assumption is refuted regardless of where $\perp$ surfaces. The same holds for $\perp$-elimination: from $w : \perp$, conclude $v : \psi$ for any v and any ψ.

$\Box$-Elimination

Rule $\Box$E

From $w : \Box\varphi$ and wRv, conclude $v : \varphi$.

$$\frac{w : \Box\varphi \qquad wRv}{v : \varphi}\ \Box\mathrm{E}$$

- If φ is necessary at w, and v is accessible from w, then φ holds at v.
- No sub-proof needed, no freshness condition.
- Directly reflects: $\mathcal{M}, w \models \Box\varphi$ iff for all v with Rwv, $\mathcal{M}, v \models \varphi$.

$\Box$-Introduction

Rule $\Box$I

Open a sub-proof assuming wRv with **fresh** v. Derive $v : \varphi$. Close and conclude $w : \Box\varphi$.

$$\dfrac{\begin{array}{c} wRv \quad (\text{fresh } v) \\ \vdots \\ v : \varphi \end{array}}{w : \Box\varphi}\ \Box\mathrm{I}$$

Idea: to show φ is necessary at w, show it holds at an *arbitrary* accessible world.

Since v is fresh, the derivation applies to *all* accessible worlds.

$\Box$I — What May Be Used Inside

Inside the $\Box$I sub-proof, you may use:

- The assumption wRv itself
- Any formula obtained by $\Box$E from a *boxed* formula outside (e.g. from $w : \Box\psi$, derive $v : \psi$)
- Anything derivable from the above

You may NOT

Import unboxed formulae from outside into a new label.
If $w : p$ is available outside, you *cannot* write $v : p$ inside: p might be true at w without being true at v.

$\Diamond$-Introduction

Rule $\Diamond$I

From wRv and $v : \varphi$, conclude $w : \Diamond\varphi$.

$$\frac{wRv \qquad v : \varphi}{w : \Diamond\varphi}\ \Diamond\mathrm{I}$$

If some accessible world satisfies φ, then φ is possible. No sub-proof needed.

$\Diamond$-Elimination

Rule $\Diamond$E

From $w : \Diamond\varphi$, assume fresh wRv and $v : \varphi$. Derive χ not mentioning v. Close and assert χ.

$$\dfrac{w : \Diamond\varphi \qquad \begin{array}{c} wRv,\ v : \varphi \quad (\text{fresh } v) \\ \vdots \\ \chi \quad (v \notin \chi) \end{array}}{\chi}\ \Diamond\text{E}$$

Idea: $\Diamond\varphi$ says *some* accessible world satisfies φ; give it a fresh name, reason about it, derive a conclusion independent of the witness.

The $\Box/\Diamond$ and $\forall/\exists$ Analogy

	Universal	Existential
First-order logic	$\forall$	$\exists$
Modal logic	$\Box$	$\Diamond$
Introduction	fresh eigenvariable / world	witness term / world
Elimination	instantiate to any term / world	assume fresh witness

The labelled ND system makes this analogy *exact*: the rules are structurally identical.

Summary of Modal Rules

Rule	Premises	Conclusion
$\Box$E	$w : \Box\varphi$, wRv	$v : \varphi$
$\Box$I	sub-proof: fresh wRv, derive $v : \varphi$	$w : \Box\varphi$
$\Diamond$I	wRv, $v : \varphi$	$w : \Diamond\varphi$
$\Diamond$E	$w : \Diamond\varphi$; sub-proof: fresh wRv, $v : \varphi$, derive χ	χ ($v \notin \chi$)

None of the modal rules is branching. Branching comes only from the propositional rules.

Why the Freshness Condition Matters

Soundness of $\Box$I: because v is fresh, the derivation of $v : \varphi$ applies to a *generic* accessible world $\Rightarrow$ it applies to *all* accessible worlds.

What goes wrong without freshness — an incorrect "proof" of $\Diamond p \vdash_K \Box p$:

1	$w : \Diamond p$	P
2	$wRv, \quad v : p$	fresh v
3	wRv	**ERROR: v not fresh!**
4	$v : p$	R, 2
5	$w : \Box p$	$\Box$I, 3—4
6	$w : \Box p$	$\Diamond$E, 1, 2—5

$\Diamond p \vdash_K \Box p$ is semantically *invalid* $\Rightarrow$ the freshness violation produced an invalidity.

Three Common Mistakes

1. **Importing unboxed formulae into a new world.**
   If $w : p$ is outside, you cannot write $v : p$ inside a $\Box$I sub-proof. Only *boxed* formulae cross worlds (via $\Box$E).

2. **Reusing a non-fresh label in $\Box$I.**
   The label v must not occur earlier in the proof. Reusing a witness from $\Diamond$E destroys the generality.

3. **Applying necessitation to a non-theorem.**
   From $\varphi \vdash_K \varphi$ one *cannot* conclude $\varphi \vdash_K \Box\varphi$. Necessitation only applies to formulae derived from *zero* premises.

Example 1: $\Box p \wedge \Box q \vdash_K \Box(p \wedge q)$

1	$w : \Box p \wedge \Box q$	P
2	$w : \Box p$	$\wedge$E, 1
3	$w : \Box q$	$\wedge$E, 1
4	$\quad wRv$	fresh v
5	$\quad v : p$	$\Box$E, 2, 4
6	$\quad v : q$	$\Box$E, 3, 4
7	$\quad v : p \wedge q$	$\wedge$I, 5, 6
8	$w : \Box(p \wedge q)$	$\Box$I, 4—7

Open a $\Box$I sub-proof at fresh v; use $\Box$E to bring p and q down; form the conjunction at v; close.

Example 2 (converse): $\Box(p \land q) \vdash_K \Box p \land \Box q$

1	$w : \Box(p \land q)$	P
2	$\quad wRv$	fresh v
3	$\quad v : p \land q$	$\Box$E, 1, 2
4	$\quad v : p$	$\land$E, 3
5	$w : \Box p$	$\Box$I, 2—4
6	$\quad wRu$	fresh u
7	$\quad u : p \land q$	$\Box$E, 1, 6
8	$\quad u : q$	$\land$E, 7
9	$w : \Box q$	$\Box$I, 6—8
10	$w : \Box p \land \Box q$	$\land$I, 5, 9

Two separate $\Box$I sub-proofs, each with its own fresh label.

Example 3: The Distribution Axiom — $\Box(p \to q) \vdash_K \Box p \to \Box q$

1	$w : \Box(p \to q)$	P
2	$\quad w : \Box p$	H
3	$\quad\quad wRv$	fresh v
4	$\quad\quad v : p \to q$	$\Box$E, 1, 3
5	$\quad\quad v : p$	$\Box$E, 2, 3
6	$\quad\quad v : q$	$\to$E, 4, 5
7	$\quad w : \Box q$	$\Box$I, 3—6
8	$w : \Box p \to \Box q$	$\to$I, 2—7

Nested sub-proofs: $\to$-intro on the outside, $\Box$I on the inside.

Example 4: $\Box((p \wedge q) \to r), \Box p \vdash_K \Box(q \to r)$

1	$w : \Box((p \wedge q) \to r)$	P
2	$w : \Box p$	P
3	$\quad wRv$	fresh v
4	$\quad v : (p \wedge q) \to r$	$\Box$E, 1, 3
5	$\quad v : p$	$\Box$E, 2, 3
6	$\quad\quad v : q$	H
7	$\quad\quad v : p \wedge q$	$\wedge$I, 5, 6
8	$\quad\quad v : r$	$\to$E, 4, 7
9	$\quad v : q \to r$	$\to$I, 6—8
10	$w : \Box(q \to r)$	$\Box$I, 3—9

Example 5: Working with $\Diamond$ — $\Diamond(p \wedge q) \vdash_K \Diamond p \wedge \Diamond q$

1	$w : \Diamond(p \wedge q)$	P
2	$\quad wRv, \quad v : p \wedge q$	fresh v
3	$\quad v : p$	$\wedge$E, 2
4	$\quad v : q$	$\wedge$E, 2
5	$\quad w : \Diamond p$	$\Diamond$I, 2, 3
6	$\quad w : \Diamond q$	$\Diamond$I, 2, 4
7	$\quad w : \Diamond p \wedge \Diamond q$	$\wedge$I, 5, 6
8	$w : \Diamond p \wedge \Diamond q$	$\Diamond$E, 1, 2—7

$\Diamond$E opens a sub-proof; $\Diamond$I closes back to w.

Example 6: Combining $\Box$ and $\Diamond$ — $\Box(p \to q) \vdash_K \Diamond p \to \Diamond q$

1	$w : \Box(p \to q)$	P
2	$\quad w : \Diamond p$	H
3	$\quad\quad wRv, \quad v : p$	fresh v
4	$\quad\quad v : p \to q$	$\Box$E, 1, 3
5	$\quad\quad v : q$	$\to$E, 4, 3
6	$\quad\quad w : \Diamond q$	$\Diamond$I, 3, 5
7	$\quad w : \Diamond q$	$\Diamond$E, 2, 3—6
8	$w : \Diamond p \to \Diamond q$	$\to$I, 2—7

Example 7: $\Diamond$E with Reductio — $\Diamond p \wedge \Box\neg p \vdash_K \bot$

1	$w : \Diamond p \wedge \Box \neg p$	P
2	$w : \Diamond p$	$\wedge$E, 1
3	$w : \Box \neg p$	$\wedge$E, 1
4	$\quad wRv,\ \ v : p$	fresh v
5	$\quad v : \neg p$	$\Box$E, 3, 4
6	$\quad v : \bot$	$\neg$E, 4, 5
7	$w : \bot$	$\Diamond$E, 2, 4—6

The $\Diamond$ witness provides p at v; $\Box$E provides $\neg p$ at the same world. Contradiction.

The Definitional Equivalence: $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$

Left-to-right: $\Diamond\varphi \vdash_K \neg\Box\neg\varphi$

1	$w : \Diamond\varphi$	P
2	$\quad w : \Box\neg\varphi$	H
3	$\quad\quad wRv,\quad v : \varphi$	fresh v
4	$\quad\quad v : \neg\varphi$	$\Box$E, 2, 3
5	$\quad\quad v : \bot$	$\neg$E, 3, 4
6	$\quad w : \bot$	$\Diamond$E, 1, 3—5
7	$w : \neg\Box\neg\varphi$	$\neg$I, 2—6

Assume $\Box\neg\varphi$ for $\neg$-intro; inside, $\Diamond$E gives φ and $\neg\varphi$ at the same world.

The Definitional Equivalence (continued)

Right-to-left: $\neg\Box\neg\varphi \vdash_K \Diamond\varphi$

1	$w : \neg\Box\neg\varphi$	P
2	$\quad w : \neg\Diamond\varphi$	H
3	$\quad\quad wRv$	fresh v
4	$\quad\quad\quad v : \varphi$	H
5	$\quad\quad\quad w : \Diamond\varphi$	$\Diamond$I, 3, 4
6	$\quad\quad\quad w : \bot$	$\neg$E, 2, 5
7	$\quad\quad v : \neg\varphi$	$\neg$I, 4—6
8	$\quad w : \Box\neg\varphi$	$\Box$I, 3—7
9	$\quad w : \bot$	$\neg$E, 1, 8
10	$w : \Diamond\varphi$	RAA, 2—9

Axiomatic Presentation of K

K is the smallest logic containing:

1. All propositional tautologies
2. The **distribution axiom**:

$$\vdash_K \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

3. The **necessitation rule**: if $\vdash_K \varphi$, then $\vdash_K \Box\varphi$

closed under modus ponens.

We already derived the distribution axiom in Example 3.

Caution about Necessitation

The necessitation rule is NOT a conditional

From $p \vdash_K p$ one *cannot* conclude $p \vdash_K \Box p$.

The rule only applies to formulae derived from **zero premises**:

$$\text{If } \vdash_K \varphi \quad \text{then} \quad \vdash_K \Box \varphi$$

All logical theorems receive the highest modal status: a tautology is true at every world in every model, so a fortiori at every accessible world.

What would Descartes have thought about that?

Necessitation: Model-Theoretic Proof

Theorem

If $\models_K \varphi$*, then* $\models_K \Box\varphi$*.*

Proof.

Suppose for contradiction that $\models_K \varphi$ but $\not\models_K \Box\varphi$.

Then there exist $\mathcal{M}, w$ such that $\mathcal{M}, w \not\models \Box\varphi$.

So there exists w' with Rww' and $\mathcal{M}, w' \not\models \varphi$.

But $\models_K \varphi$ means φ is true at *every* index in *every* model.

In particular $\mathcal{M}, w' \models \varphi$. Contradiction. □

System T (or M): The Necessary Is Actual

Axiom T (or M)

$\vdash_T \Box\varphi \to \varphi$

Frame condition: reflexivity ($\forall w.\ Rww$).

As a proof rule ($\Box$E-M): From $w : \Box\varphi$, conclude $w : \varphi$.

$$\frac{w : \Box\varphi}{w : \varphi}\ \Box\text{E-}M$$

No accessibility statement needed (reflexivity guarantees wRw).

Boxes Can Be Eliminated!

In T, any number of nested boxes can be stripped away:

1	$w : \Box\Box p$	P
2	$w : \Box p$	$\Box$E-M, 1
3	$w : p$	$\Box$E-M, 2

A Theorem of T: $\varphi \to \Diamond\varphi$

In T, whatever is actual is possible.

1	$w : \varphi$	H
2	$w : \neg\Diamond\varphi$	H
3	$w : \Box\neg\varphi$	Def. $\Diamond$, 2
4	$w : \neg\varphi$	$\Box$E-M, 3
5	$w : \bot$	$\neg$E, 1, 4
6	$w : \neg\neg\Diamond\varphi$	$\neg$I, 2—5
7	$w : \Diamond\varphi$	$\neg\neg$E, 6
8	$w : \varphi \to \Diamond\varphi$	$\to$I, 1—7

Key step: line 4 uses $\Box$E-M (unavailable in K!).

Iterated Modalities

The system T says the necessary is actual. But:

- Is the necessary *necessarily* necessary?
- Is the actual *necessarily* possible?
- Is the possible *necessarily* possible?

$S4$: Positive Introspection

Axiom 4

$\vdash_{S4} \Box\varphi \to \Box\Box\varphi$ *The necessary is necessarily necessary.*

Frame condition: transitivity. $S4 = T +$ axiom 4.

Transitivity rule: From wRv and vRu, conclude wRu.

Proof of $\Box p \vdash_{S4} \Box\Box p$:

1	$w : \Box p$	P
2	wRv	fresh v
3	vRu	fresh u
4	wRu	Trans., 2, 3
5	$u : p$	$\Box$E, 1, 4
6	$v : \Box p$	$\Box$I, 3—5
7	$w : \Box\Box p$	$\Box$I, 2—6

Without transitivity, line 4 is unavailable $\Rightarrow$ fails in K.

B and $S5$

Axiom B (Brouwer)

$\vdash_B \varphi \to \Box\Diamond\varphi$ *The actual is necessarily possible.*

Frame condition: **symmetry** ($wRv \Rightarrow vRw$).

Axiom 5

$\vdash_{S5} \Diamond\varphi \to \Box\Diamond\varphi$ *The possible is necessarily possible.*

Frame condition: **euclideanness** ($wRv \wedge wRu \Rightarrow vRu$).

$S5 = T +$ axiom 5: R is an equivalence relation.
Iterated modalities collapse: $\Box\Box\varphi \leftrightarrow \Box\varphi$ and $\Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$.

Summary of Frame Conditions

System	Axiom	Frame condition
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	none
D	$\Box\varphi \to \Diamond\varphi$	seriality
T (M)	$\Box\varphi \to \varphi$	reflexivity
$S4$	$\Box\varphi \to \Box\Box\varphi$	refl. + transitivity
B	$\varphi \to \Box\Diamond\varphi$	refl. + symmetry
$S5$	$\Diamond\varphi \to \Box\Diamond\varphi$	equivalence relation

Deontic Logic: System D

New operator O ("it is obligatory that"), satisfying the rules of K.

Dual operators:

- $P\varphi =_{\text{Def}} \neg O \neg \varphi$: "it is *permitted* that φ"
- $F\varphi =_{\text{Def}} O \neg \varphi$: "it is *forbidden* that φ"

Axiom D

$O\varphi \rightarrow P\varphi$ (equivalently: $O\varphi \rightarrow \neg O \neg \varphi$)

What is obligatory is permitted. (Remember Kant: "ought implies can.")

Frame condition: **seriality** ($\forall w \exists v.\ wRv$).

Why not axiom T?
$O\varphi \to \varphi$ would say: what is obligatory is actual. Plainly false!
Deontic logic uses D (seriality), not T (reflexivity).

Necessitation in deontic logic.
All logical tautologies are obligatory: $\vdash_D O(p \vee \neg p)$.

Is this correct?

- Pro: logical truths are trivially fulfilled and cost nothing to obey.
- Con: it seems odd that logical necessities should count as moral obligations.

Tense Logic

Two pairs of operators:

- Future: $G\varphi$ ("always going to be"), $F\varphi$ ("will be")
- Past: $H\varphi$ ("always has been"), $P\varphi$ ("has been")

As usual: $F\varphi =_{\text{Def}} \neg G \neg \varphi$ and $P\varphi =_{\text{Def}} \neg H \neg \varphi$

Why axiom T fails: $G\varphi \rightarrow \varphi$ says "if φ will always be true, then φ is true now." But the "strictly later than" relation is *irreflexive*: the present is not in its own future.

Tense logics are extensions of K: necessitation and distribution hold for both G and H.

Laws for Tenses

Interaction principles: the future and past operators are linked.

Axiom GP

$\varphi \to GP\varphi$

"If φ is true now, it will always have been true."

Axiom HF

$\varphi \to HF\varphi$

"If φ is true now, it always was going to be true."

Operator: $B_i\varphi$ ("agent i believes that φ").
Each B_i satisfies the rules of K.

Logical omniscience

If φ is a logical truth, then $B_i\varphi$ for every agent i.
Descriptively unrealistic. Normatively defensible?

Axiom D for beliefs: $B_i\varphi \to \neg B_i\neg\varphi$
If an agent believes φ, the agent does not also believe $\neg\varphi$. (Consistency.)

Knowledge vs. Belief

Axiom T for beliefs? $B_i\varphi \to \varphi$ — No!
As Plato remarked: beliefs can be false.

But **knowledge is factive**: if you know φ, then φ is true.

$$K_i\varphi \to \varphi \qquad \text{(axiom } T \text{ for knowledge)}$$

Positive introspection (axiom 4):

$$K_i\varphi \to K_iK_i\varphi \qquad\qquad B_i\varphi \to B_iB_i\varphi$$

Negative introspection (axiom 5):

$$\neg K_i\varphi \to K_i\neg K_i\varphi \qquad\qquad \neg B_i\varphi \to B_i\neg B_i\varphi$$

Epistemic logic with both $\Rightarrow$ system $S5$ for knowledge.

Summary

1. **Labelled natural deduction** for K: four modal rules ($\Box$E, $\Box$I, $\Diamond$I, $\Diamond$E), structurally parallel to $\forall/\exists$ in FOL.

2. **Distribution axiom** + **necessitation rule** characterise K.

3. **Extensions**: T (reflexivity), $S4$ (transitivity), B (symmetry), $S5$ (equivalence), D (seriality).

4. **Applications**: deontic logic, tense logic, epistemic logic.