Modal Logic II

A Natural Deduction System for K and Extensions

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Part I
A Labelled Natural Deduction System for K

1 From Tableaux to Proofs

In Modal Logic I, we studied the *semantics* of K using Kripke models and the *tableau method* to test validity and build counter-models. We now turn to the *proof theory* of K: a formal system in which valid arguments can be *derived* step by step.

We present a **labelled natural deduction** system. The central idea is simple: every formula in a proof carries a **world label** (a name for a possible world), and **accessibility assertions** are handled explicitly. This approach has two key advantages:

- It connects directly to the Kripke semantics and to the tableaux we already know: the labels are the indices $1, 2, 3, \ldots$ from the tableau rules, and the accessibility pairs iRj are exactly the same.
- The rules for $\Box$ and $\Diamond$ are structurally parallel to the rules for $\forall$ and $\exists$ in first-order logic, making the analogy between quantifiers and modal operators precise.

2 Labelled Formulae

Definition 2.1 (Labelled formula)**.** A **labelled formula** is an expression of the form

$$w : \varphi$$

where w is a **world label** (typically written w, v, u, or simply $1, 2, 3$) and φ is a formula of modal propositional logic. It is read: "φ holds at world w."

Definition 2.2 (Accessibility statement)**.** An **accessibility statement** is an expression of the form

$$wRv$$

where w and v are world labels. It is read: "v is accessible from w."

A proof in our system consists of a sequence of labelled formulae and accessibility statements, each justified either as a hypothesis, as an assumption opening a sub-proof, or as following from earlier lines by a rule.

3 The Rules

3.1 Propositional Rules

All the standard propositional natural deduction rules (for $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\bot$) carry over unchanged, **provided all the formulae involved share the same world label**. In other words, propositional reasoning operates *within a single world*.

For instance, $\rightarrow$-introduction works as follows: if, by assuming $w : \varphi$, we can derive $w : \psi$, then we may close the sub-proof and conclude $w : \varphi \rightarrow \psi$.

Important constraint. Propositional rules cannot mix labels. One cannot, for instance, combine $w : \varphi$ and $v : \varphi \rightarrow \psi$ to conclude $v : \psi$: the modus ponens must happen at a single world.

Remark 3.1 (Exception: negation introduction and *ex falso*). There is one important exception to the constraint above. The rule of negation introduction ($\neg$I) permits labels to be mixed. If assuming $v : \varphi$ leads to a contradiction at *any* world—say $w : \bot$ with $w \neq v$—then we may discharge the assumption and conclude $v : \neg\varphi$:

1	$\quad v : \varphi$	H
2	$\quad \vdots$	
3	$\quad w : \bot$	$(w \neq v)$
4	$v : \neg\varphi$	$\neg$I, 1—3

The reason is that $\bot$ is not world-relative: if any world in a model harbours a contradiction, the entire model is impossible, so the assumption $v : \varphi$ is refuted regardless of where the contradiction surfaces. The same applies to $\bot$-elimination (*ex falso*): from $w : \bot$ one may conclude $v : \psi$ for any label v and any formula ψ.

We shall see a concrete instance of this pattern in the proof that $\neg\Box\neg\varphi \vdash_K \Diamond\varphi$ (Section **??**).

3.2 $\Box$-Elimination

Rule $\Box$E ($\Box$-elimination). From $w : \Box\varphi$ and wRv, conclude $v : \varphi$.

$$\frac{w : \Box\varphi \qquad wRv}{v : \varphi}\,\Box\mathrm{E}$$

This is the simplest modal rule: if φ is necessary at w, and v is accessible from w, then φ holds at v. It directly reflects the semantic clause $\mathcal{M}, w \models \Box\varphi$ iff for all v with Rwv, $\mathcal{M}, v \models \varphi$.

The rule requires no sub-proof, no freshness condition on labels, and no side conditions. It can be applied whenever the two premises are available.

3.3 $\Box$-Introduction

Rule $\Box$I ($\Box$-introduction). Open a sub-proof by assuming a fresh accessibility statement wRv (where v is a new label that does not occur anywhere earlier in the proof). Within this sub-proof, derive $v : \varphi$. Then close the sub-proof and conclude $w : \Box\varphi$.

$$\frac{\begin{array}{l} wRv \quad \text{(fresh } v\text{)} \\ \quad\vdots \\ v : \varphi \end{array}}{w : \Box\varphi}\,\Box\mathrm{I}$$

The idea: to show that φ is necessary at w, we show that φ holds at an *arbitrary* world accessible from w. Since v is fresh, it represents a "generic" accessible world; whatever we prove about it holds for all accessible worlds.

Remark 3.2. The freshness condition on v is exactly parallel to the freshness condition on the eigenvariable in $\forall$-introduction in first-order logic. In $\forall$-intro, one introduces an arbitrary constant a, proves $\varphi(a)$, and concludes $\forall x\,\varphi(x)$; here, one introduces an arbitrary accessible world v, proves $v : \varphi$, and concludes $w : \Box\varphi$.

What may be used inside the sub-proof. Inside the $\Box$I sub-proof, you may use:

- the assumption wRv itself,
- any formula obtained by applying $\Box$E to a boxed formula from outside the sub-proof (i.e. if $w : \Box\psi$ is available outside, you may derive $v : \psi$ inside by $\Box$E),
- anything derivable from the above by the other rules.

You may *not* import unboxed formulae from outside the sub-proof into a new label. For instance, if $w : p$ is available outside, you cannot simply write $v : p$ inside: p might be true at w without being true at v.

3.4 $\Diamond$-Introduction

Rule $\Diamond$I ($\Diamond$-introduction). From wRv and $v : \varphi$, conclude $w : \Diamond\varphi$.

$$\frac{wRv \qquad v : \varphi}{w : \Diamond\varphi}\ \Diamond\mathrm{I}$$

This is straightforward: if some accessible world satisfies φ, then φ is possible. No sub-proof is needed.

3.5 $\Diamond$-Elimination

Rule $\Diamond$E ($\Diamond$-elimination). From $w : \Diamond\varphi$, open a sub-proof assuming wRv and $v : \varphi$ (where v is a fresh label). Derive a conclusion χ that does not mention v. Then close the sub-proof and assert χ.

$$\frac{w : \Diamond\varphi \qquad \begin{array}{c} wRv,\ v : \varphi \quad (\text{fresh } v) \\ \vdots \\ \chi \quad (v \text{ does not occur in } \chi) \end{array}}{\chi}\ \Diamond\mathrm{E}$$

The idea: $\Diamond\varphi$ tells us that *some* accessible world satisfies φ, but we do not know which one. We give it a fresh name v, reason about it, and derive a conclusion that does not depend on the particular identity of v.

Remark 3.3. This is exactly parallel to $\exists$-elimination in first-order logic: from $\exists x\,\varphi(x)$, assume $\varphi(a)$ for a fresh constant a, derive a conclusion not mentioning a, then discharge. The structural analogy between $\Box/\Diamond$ and $\forall/\exists$ is now complete:

	Universal	**Existential**
First-order logic	$\forall$	$\exists$
Modal logic	$\Box$	$\Diamond$
Introduction	fresh eigenvariable / fresh world	witness term / witness world
Elimination	instantiate to any term / world	assume fresh witness

3.6 Summary of the Modal Rules

Rule	**Premises**	**Conclusion**
$\Box$E	$w : \Box\varphi$, wRv	$v : \varphi$
$\Box$I	sub-proof: assume fresh wRv, derive $v : \varphi$	$w : \Box\varphi$
$\Diamond$I	wRv, $v : \varphi$	$w : \Diamond\varphi$
$\Diamond$E	$w : \Diamond\varphi$; sub-proof: assume fresh wRv, $v : \varphi$, derive χ not mentioning v	χ ($v \notin \chi$)

3.7 Why the Rules Are Sound

Each rule preserves truth in Kripke models. The propositional rules and $\Box$E/$\Diamond$I are straightforward; the two sub-proof rules ($\Box$I and $\Diamond$E) require a brief argument.

Soundness of $\Box$I. Suppose that, in every Kripke model, whenever wRv holds we can derive $v : \varphi$ using only what is available inside the sub-proof. We must show that $w : \Box\,\varphi$ holds, i.e. that $\mathcal{M}, v \models \varphi$ for *every* v with wRv. The **freshness condition** is what makes this work: because v does not appear anywhere else, the derivation of $v : \varphi$ cannot depend on any specific property of v. It applies to an *arbitrary* accessible world, so it applies to all of them.

What goes wrong without freshness. Consider the following incorrect "proof" of $\Diamond\, p \vdash_K \Box\, p$:

1	$w : \Diamond\, p$	P
2	$\quad wRv,\ \ v : p$	fresh v
3	$\quad\quad wRv$	**ERROR: v not fresh!**
4	$\quad\quad v : p$	R, 2
5	$\quad w : \Box\, p$	$\Box$I, 3—4
6	$w : \Box\, p$	$\Diamond$E, 1, 2—5

The error is at line 3: v already occurs in line 2, so it is *not* fresh. The $\Box$I sub-proof at lines 3–4 "proves" that p holds at v, but v was introduced as a specific witness for $\Diamond\, p$—not as a generic accessible world. The "conclusion" $\Diamond\, p \vdash_K \Box\, p$ is semantically invalid ($\Diamond\, p$ does not entail $\Box\, p$), confirming that the freshness violation produced nonsense.

Soundness of $\Diamond$E. Suppose $w : \Diamond\,\varphi$ holds and that, by assuming wRv and $v : \varphi$ for a fresh v, we can derive χ (not mentioning v). We must show that χ holds. Since $\Diamond\,\varphi$ is true at w, there exists *some* world v_0 with wRv_0 and $\mathcal{M}, v_0 \models \varphi$. The derivation of χ from wRv and $v : \varphi$ applies in particular to v_0; since χ does not mention v, the conclusion is independent of which witness was chosen.

3.8 Common Mistakes

Three errors account for the vast majority of incorrect proofs in modal natural deduction. Each corresponds to a violation of the scoping discipline.

Mistake 1: Importing unboxed formulae into a new world. If $w : p$ is available outside a $\Box$I sub-proof, one *cannot* write $v : p$ inside. Only boxed formulae can cross from w to v (via $\Box$E). In Kripke models, p may be true at w without being true at an accessible world v.

Mistake 2: Reusing a label in $\Box$I. The label v in a $\Box$I sub-proof must be *fresh*: it must not occur anywhere earlier in the proof. Reusing a label that already carries specific information (e.g. from a $\Diamond$E assumption) destroys the generality required for the universal conclusion.

Mistake 3: Applying necessitation to a non-theorem. From $\varphi \vdash_K \varphi$ one *cannot* conclude $\varphi \vdash_K \Box\,\varphi$. The necessitation rule only applies to formulae derived from *zero* premises. In the labelled system, this means: one may only apply $\Box$I when the sub-proof uses no undischarged assumptions other than the accessibility hypothesis wRv itself.

4 Worked Examples

4.1 $\Box p \wedge \Box q \vdash_K \Box(p \wedge q)$

1	$w : \Box p \wedge \Box q$	P
2	$w : \Box p$	$\wedge$E, 1
3	$w : \Box q$	$\wedge$E, 1
4	$\quad wRv$	fresh v
5	$\quad v : p$	$\Box$E, 2, 4
6	$\quad v : q$	$\Box$E, 3, 4
7	$\quad v : p \wedge q$	$\wedge$I, 5, 6
8	$w : \Box(p \wedge q)$	$\Box$I, 4—7

Explanation:

1. Lines 2–3: extract $\Box p$ and $\Box q$ from the conjunction (at world w).
2. Line 4: open a sub-proof for $\Box$I by assuming that v is an arbitrary world accessible from w.
3. Lines 5–6: use $\Box$E to bring p and q down to world v.
4. Line 7: form the conjunction $p \wedge q$ at v (propositional rule, same world).
5. Line 8: close the sub-proof and conclude $\Box(p \wedge q)$ at w by $\Box$I.

4.2 The Converse: $\Box(p \wedge q) \vdash_K \Box p \wedge \Box q$

Together with the previous proof, this establishes that $\Box p \wedge \Box q$ and $\Box(p \wedge q)$ are logically equivalent in K: necessity distributes over conjunction in both directions.

1	$w : \Box(p \wedge q)$	P
2	$\quad wRv$	fresh v
3	$\quad v : p \wedge q$	$\Box$E, 1, 2
4	$\quad v : p$	$\wedge$E, 3
5	$w : \Box p$	$\Box$I, 2—4
6	$\quad wRu$	fresh u
7	$\quad u : p \wedge q$	$\Box$E, 1, 6
8	$\quad u : q$	$\wedge$E, 7
9	$w : \Box q$	$\Box$I, 6—8
10	$w : \Box p \wedge \Box q$	$\wedge$I, 5, 9

Explanation: Two separate $\Box$I sub-proofs, one for $\Box p$ (lines 2–4) and one for $\Box q$ (lines 6–8), each opening its own fresh label. Note that u in the second sub-proof must be distinct from v, since v was already used; in practice, any new letter will do.

4.3 $\Box(p \to q) \vdash_K \Box p \to \Box q$ — The Distribution Axiom

1	$w : \Box(p \to q)$	P
2	$\quad w : \Box p$	H
3	$\quad\quad wRv$	fresh v
4	$\quad\quad v : p \to q$	$\Box$E, 1, 3
5	$\quad\quad v : p$	$\Box$E, 2, 3
6	$\quad\quad v : q$	$\to$E, 4, 5
7	$\quad w : \Box q$	$\Box$I, 3—6
8	$w : \Box p \to \Box q$	$\to$I, 2—7

Explanation:

1. Line 2: assume $\Box p$ at w (for $\to$-introduction).
2. Lines 3–6: open a $\Box$I sub-proof with fresh v. At v, both $p \to q$ and p are available via $\Box$E; modus ponens gives q.
3. Line 7: close the $\Box$I sub-proof: $\Box q$ at w.
4. Line 8: close the $\to$-intro sub-proof: $\Box p \to \Box q$ at w.

Remark 4.1. This proof is a derivation of the distribution axiom K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. The nested sub-proofs (one for $\to$-intro, one for $\Box$I inside it) are characteristic of proofs that mix propositional and modal reasoning.

4.4 $\Box((p \wedge q) \to r),\ \Box p \vdash_K \Box(q \to r)$

1	$w : \Box((p \wedge q) \to r)$	P
2	$w : \Box p$	P
3	$\quad wRv$	fresh v
4	$\quad v : (p \wedge q) \to r$	$\Box$E, 1, 3
5	$\quad v : p$	$\Box$E, 2, 3
6	$\quad\quad v : q$	H
7	$\quad\quad v : p \wedge q$	$\wedge$I, 5, 6
8	$\quad\quad v : r$	$\to$E, 4, 7
9	$\quad v : q \to r$	$\to$I, 6—8
10	$w : \Box(q \to r)$	$\Box$I, 3—9

Explanation: We open a $\Box$I sub-proof at fresh v (line 3), bring down p and $(p \wedge q) \to r$ by $\Box$E (lines 4–5), then use an inner $\to$-intro sub-proof (lines 6–8) to derive $q \to r$ at v.

4.5 Working with $\Diamond$: $\Diamond(p \wedge q) \vdash_K \Diamond p \wedge \Diamond q$

This proof illustrates $\Diamond$-elimination and $\Diamond$-introduction.

1	$w : \Diamond(p \wedge q)$	P
2	$\mid\ wRv,\quad v : p \wedge q$	fresh v
3	$\mid\ v : p$	$\wedge$E, 2
4	$\mid\ v : q$	$\wedge$E, 2
5	$\mid\ w : \Diamond p$	$\Diamond$I, 2, 3
6	$\mid\ w : \Diamond q$	$\Diamond$I, 2, 4
7	$\mid\ w : \Diamond p \wedge \Diamond q$	$\wedge$I, 5, 6
8	$w : \Diamond p \wedge \Diamond q$	$\Diamond$E, 1, 2—7

Explanation:

1. Line 1: we are given $\Diamond(p \wedge q)$ at w.
2. Line 2: open a $\Diamond$E sub-proof: assume a fresh world v accessible from w where $p \wedge q$ holds.
3. Lines 3–4: split the conjunction at v.
4. Lines 5–6: from p at v and wRv, conclude $\Diamond p$ at w by $\Diamond$I. Similarly for q.
5. Line 7: form the conjunction $\Diamond p \wedge \Diamond q$ at w.
6. Line 8: close the sub-proof by $\Diamond$E. The conclusion (line 7) mentions only w, not v, so the freshness condition is satisfied.

Remark 4.2. Observe how direct the $\Diamond$-rules are. In the strict sub-proof system, this proof requires going through $\neg\Box\neg$ with nested negation-introduction sub-proofs. Here, $\Diamond$ is treated as a first-class operator.

4.6 Combining $\Box$ and $\Diamond$: $\Box(p \to q) \vdash_K \Diamond p \to \Diamond q$

1	$w : \Box(p \to q)$	P
2	$\mid\ w : \Diamond p$	H
3	$\mid\ \mid\ wRv,\quad v : p$	fresh v
4	$\mid\ \mid\ v : p \to q$	$\Box$E, 1, 3
5	$\mid\ \mid\ v : q$	$\to$E, 4, 3
6	$\mid\ \mid\ w : \Diamond q$	$\Diamond$I, 3, 5
7	$\mid\ w : \Diamond q$	$\Diamond$E, 2, 3—6
8	$w : \Diamond p \to \Diamond q$	$\to$I, 2—7

Explanation: The overall goal is to prove a conditional (line 8), so we assume its antecedent $\Diamond p$ at w (line 2). To extract information from $\Diamond p$, we open a $\Diamond$E sub-proof at fresh v (line 3). Inside, $\Box$E gives us $p \to q$ at v (line 4), modus ponens gives q at v (line 5), and $\Diamond$I yields $\Diamond q$ at w (line 6). Since $\Diamond q$ at w does not mention v, the $\Diamond$E sub-proof closes.

4.7 $\Diamond$E with Reductio: $\Diamond p \wedge \Box\neg p \vdash_K \bot$

The previous $\Diamond$E examples were "positive": the witness fed into $\Diamond$I. Here we show how $\Diamond$E interacts with *negation*, producing a contradiction.

$$\begin{array}{lll}
1 & w : \Diamond p \wedge \Box \neg p & \text{P} \\ \hline
2 & w : \Diamond p & \wedge\text{E, 1} \\
3 & w : \Box \neg p & \wedge\text{E, 1} \\
4 & \quad \mid\ wRv,\ \ v : p & \text{fresh } v \\
5 & \quad \mid\ v : \neg p & \Box\text{E, 3, 4} \\
6 & \quad \mid\ v : \bot & \neg\text{E, 4, 5} \\
7 & w : \bot & \Diamond\text{E, 2, 4—6}
\end{array}$$

Explanation:

1. Lines 2–3: split the conjunction.
2. Line 4: open a $\Diamond$E sub-proof: assume a fresh world v accessible from w where p holds.
3. Line 5: since $\Box\neg p$ holds at w and wRv, we get $\neg p$ at v by $\Box$E.
4. Line 6: p and $\neg p$ at the same world: contradiction.
5. Line 7: close the sub-proof. The conclusion $w : \bot$ does not mention v, so the discharge is valid.

Remark 4.3. An immediate corollary: $\Diamond p \vdash_K \neg\Box\neg p$. Assume $\Diamond p$ and $\Box\neg p$; the proof above yields $\bot$; by $\neg$-introduction, discharge $\Box\neg p$ and conclude $\neg\Box\neg p$. This is one half of the equivalence $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$.

4.8 The Definitional Equivalence: $\vdash_K \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$

Our labelled system treats $\Diamond$ as a primitive operator with its own rules. But the standard definition $\Diamond\varphi =_{\text{Def}} \neg\Box\neg\varphi$ should be derivable. We show both directions.

Left-to-right: $\Diamond\varphi \vdash_K \neg\Box\neg\varphi$.

$$\begin{array}{lll}
1 & w : \Diamond \varphi & \text{P} \\ \hline
2 & \quad \mid\ w : \Box \neg \varphi & \text{H} \\
3 & \quad \mid\ \quad \mid\ wRv,\ \ v : \varphi & \text{fresh } v \\
4 & \quad \mid\ \quad \mid\ v : \neg\varphi & \Box\text{E, 2, 3} \\
5 & \quad \mid\ \quad \mid\ v : \bot & \neg\text{E, 3, 4} \\
6 & \quad \mid\ w : \bot & \Diamond\text{E, 1, 3—5} \\
7 & w : \neg\Box\neg\varphi & \neg\text{I, 2—6}
\end{array}$$

Right-to-left: $\neg\Box\neg\varphi \vdash_K \Diamond\varphi$**.** This direction requires the classical rule of double negation elimination (or, equivalently, reductio ad absurdum in its classical form).

1	$w : \neg\Box\neg\varphi$	P
2	$\quad w : \neg\Diamond\varphi$	H
3	$\quad\quad wRv$	fresh v
4	$\quad\quad\quad v : \varphi$	H
5	$\quad\quad\quad w : \Diamond\varphi$	$\Diamond$I, 3, 4
6	$\quad\quad\quad w : \bot$	$\neg$E, 2, 5
7	$\quad\quad v : \neg\varphi$	$\neg$I, 4—6
8	$\quad w : \Box\neg\varphi$	$\Box$I, 3—7
9	$\quad w : \bot$	$\neg$E, 1, 8
10	$w : \neg\neg\Diamond\varphi$	$\neg$I, 2—9
11	$w : \Diamond\varphi$	$\neg\neg$E, 10

Explanation: We assume $\neg\Diamond\varphi$ for reductio (line 2). Inside a $\Box$I sub-proof (lines 3–7), we derive $\neg\varphi$ at v by assuming φ there and reaching contradiction via $\Diamond$I and the assumption $\neg\Diamond\varphi$. This gives $\Box\neg\varphi$ (line 8), contradicting the premise. Double negation elimination delivers $\Diamond\varphi$.

Remark 4.4. This proof justifies line 3 of the proof of $\varphi \to \Diamond\varphi$ in the system T (Section 7), where we invoked "Def. $\Diamond$" to pass from $\neg\Diamond\varphi$ to $\Box\neg\varphi$.

Part II
The Axiomatic Perspective

5 The Axiomatic Presentation of K

The natural deduction system we have presented is equivalent to the following axiomatic characterisation.

The modal logic K is the smallest logic containing:

1. All propositional tautologies (i.e. all theorems of propositional logic LP).
2. The **distribution axiom**:

$$\vdash_K \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi).$$

3. The **necessitation rule**: if $\vdash_K \varphi$ (that is, φ can be derived from zero premises), then $\vdash_K \Box\varphi$.

and closed under modus ponens.

We have already derived the distribution axiom in our natural deduction system (Section 4.2). The necessitation rule is captured by the $\Box$I rule applied with no premises other than the accessibility assumption.

Caution about necessitation. The necessitation rule may only be applied to a formula that has been derived *without any premise or assumption.* In particular, from $p \vdash_K p$ (trivially valid) one *cannot* conclude $p \vdash_K \Box p$: the derivation of p depends on the premise p itself. One may only conclude $\vdash_K \Box\varphi$ if $\vdash_K \varphi$, i.e. if φ is a *theorem.*

Philosophical remark. Since all logical theorems are necessary, all tautologies receive the highest modal status. This is a consequence of the semantic setup: a tautology is true at every world in every model, so a fortiori it is true at every accessible world. What would Descartes have thought about that?

6 Necessitation from a Model-Theoretic Perspective

Theorem 6.1. *If $\models_K \varphi$, then $\models_K \Box\varphi$.*

Proof. Suppose for contradiction that $\models_K \varphi$ but $\not\models_K \Box\varphi$. Then there exist a Kripke model $\mathcal{M}$ and an index w such that $\mathcal{M}, w \not\models \Box\varphi$. By the truth-definition for $\Box$, there exists w' with Rww' and $\mathcal{M}, w' \not\models \varphi$. But we assumed $\models_K \varphi$: φ is true at *every* index in *every* model; in particular $\mathcal{M}, w' \models \varphi$. Contradiction. □

Remark 6.1. This is a *meta-logical* result: it says that the set of logical truths of K is closed under the operation $\varphi \mapsto \Box\varphi$. It does *not* say that $\varphi \to \Box\varphi$ is valid (it is not, as we saw in Modal Logic I, Section 5.5).

Theorem 6.2 (Completeness)**.** *The labelled natural deduction system for K is* ***sound and complete****: $\vdash_K \varphi$ if and only if $\models_K \varphi$.*

We do not prove this here, but note that soundness follows from verifying that each rule preserves truth in Kripke models, and completeness can be established by a canonical model construction.

Part III
Extensions of K

We now survey the principal extensions of K, obtained by adding new axioms (equivalently, new structural conditions on the accessibility relation).

7 Alethic Modal Logic: The System T (or M)

Fundamental principle. What is necessary is actual.

Axiom T (or M).

$$\vdash_T \Box\varphi \to \varphi$$

This axiom corresponds to **reflexivity** of the accessibility relation ($\forall w.\ Rww$). In a reflexive frame, every world can see itself, so what is true at all accessible worlds is true in particular at the current world.

As a proof rule. In the labelled system, adding axiom T amounts to adding the following **□E-M rule**:

Rule □E-M. From $w : \Box\varphi$, conclude $w : \varphi$.

$$\frac{w : \Box\varphi}{w : \varphi}\ \Box\text{E-}M$$

(No accessibility statement needed.)

This is a special case of $\Box$E where the accessibility link wRw is guaranteed by reflexivity.

Boxes can be eliminated. A striking consequence: in T, any number of nested boxes can be stripped away.

1	$w : \Box\Box p$	P
2	$w : \Box p$	$\Box$E-M, 1
3	$w : p$	$\Box$E-M, 2

A theorem of T: $\varphi \to \Diamond\varphi$. In T, whatever is actual is possible. Here is the proof:

1	$w : \varphi$	H
2	$w : \neg\Diamond\varphi$	H
3	$w : \Box\neg\varphi$	Def. $\Diamond$, 2
4	$w : \neg\varphi$	$\Box$E-M, 3
5	$w : \bot$	$\neg$E, 1, 4
6	$w : \neg\neg\Diamond\varphi$	$\neg$I, 2—5
7	$w : \Diamond\varphi$	$\neg\neg$E, 6
8	$w : \varphi \to \Diamond\varphi$	$\to$I, 1—7

The key step is line 4: in system T, from $\Box\neg\varphi$ we can derive $\neg\varphi$ at the same world, which contradicts the assumption φ. This step is *not* available in K.

8 Iterated Modalities: $S4$, B, and $S5$

The system T tells us that the necessary is actual and the actual is possible. But it says nothing about *iterated* modalities: is the necessary necessarily necessary? Is the possible necessarily possible?

8.1 $S4$: Positive Introspection

Axiom 4. $\vdash_{S4} \Box\varphi \to \Box\Box\varphi$.

What is necessary is necessarily necessary. This corresponds to **transitivity** of R: if wRv and vRu, then wRu. In the labelled system, transitivity is captured by the rule:

From wRv and vRu, conclude wRu.

$S4 = T +$ axiom 4 (equivalently: R is reflexive and transitive, i.e. a preorder).

A proof in $S4$: $\Box p \vdash_{S4} \Box\Box p$. This derivation uses the transitivity rule (wRv and vRu give wRu) to propagate a boxed formula through two levels of accessibility.

1	$w : \Box p$	P
2	$\quad wRv$	fresh v
3	$\quad\quad vRu$	fresh u
4	$\quad\quad wRu$	Trans., 2, 3
5	$\quad\quad u : p$	$\Box$E, 1, 4
6	$\quad v : \Box p$	$\Box$I, 3—5
7	$w : \Box\Box p$	$\Box$I, 2—6

Explanation: The outer sub-proof (lines 2–6) introduces a fresh v accessible from w; the inner sub-proof (lines 3–5) introduces a fresh u accessible from v. The **transitivity rule** (line 4) gives wRu, which lets us apply $\Box$E to the premise $\Box p$ at w and obtain p at u. Closing the inner sub-proof gives $v : \Box p$; closing the outer one gives $w : \Box\Box p$.

Remark 8.1. Without transitivity, line 4 is unavailable: we have wRv and vRu but not wRu, so $\Box$E cannot extract p at u from $\Box p$ at w. This is precisely why $\Box p \to \Box\Box p$ fails in K (as we saw in Modal Logic I).

8.2 B: Brouwer's Axiom

Axiom B. $\vdash_B \varphi \to \Box\Diamond\varphi$.

What is actual is necessarily possible. This corresponds to **symmetry** of R: if wRv then vRw.

8.3 $S5$: Negative Introspection

Axiom 5. $\vdash_{S5} \Diamond\varphi \to \Box\Diamond\varphi$.

What is possible is necessarily possible. This corresponds to **euclideanness** of R: if wRv and wRu then vRu.

$S5 = T +$ axiom 5 (equivalently: R is an equivalence relation). In $S5$, all worlds within a cluster are fully interconnected, so iterated modalities collapse: $\Box\Box\varphi \leftrightarrow \Box\varphi$ and $\Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$.

8.4 Summary of Frame Conditions

System	**Axiom**	**Frame condition**
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	none
D	$\Box\varphi \to \Diamond\varphi$	seriality ($\forall w \exists v.\ wRv$)
T (M)	$\Box\varphi \to \varphi$	reflexivity ($\forall w.\ wRw$)
$S4$	$\Box\varphi \to \Box\Box\varphi$	reflexivity + transitivity
B	$\varphi \to \Box\Diamond\varphi$	reflexivity + symmetry
$S5$	$\Diamond\varphi \to \Box\Diamond\varphi$	equivalence relation

9 Deontic Logic: The System D

New notation. We introduce a deontic operator O ("it is obligatory that") which satisfies the same rules as $\Box$ in K, plus one additional axiom. The dual operators are:

- $P\varphi =_{\text{Def}} \neg O \neg\varphi$: "it is *permitted* that φ".
- $F\varphi =_{\text{Def}} O\neg\varphi$: "it is *forbidden* that φ".

Axiom D.

$$O\varphi \to P\varphi \qquad \text{equivalently:} \qquad O\varphi \to \neg O \neg \varphi$$

What is obligatory is permitted. (Remember Kant: "ought implies can.") This corresponds to **seriality**: every world sees at least one deontically ideal world.

Why not axiom T? The schema $O\varphi \to \varphi$ ("what is obligatory is actual") would say that obligations are always fulfilled. This is plainly false. Deontic logic uses D, not T.

Necessitation in deontic logic. Because of the necessitation rule, all logical theorems are obligatory: $\vdash_D O(p \vee \neg p)$, for instance. Is this correct? One might argue: logical truths are trivially fulfilled and cost nothing to obey, so their obligatory status is harmless. Others find it philosophically suspect that logical necessities should count as moral obligations.

10 Tense Logic

New operators. In tense logic, we introduce two pairs of modal operators:

- **Future**: $G\varphi$ ("it is always going to be the case that φ") and $F\varphi$ ("it will be the case that φ").
- **Past**: $H\varphi$ ("it has always been the case that φ") and $P\varphi$ ("it has been the case that φ").

As with $\Box$ and $\Diamond$ generally:

$$F\varphi =_{\text{Def}} \neg G \neg \varphi \qquad\qquad P\varphi =_{\text{Def}} \neg H \neg \varphi$$

Why axiom T fails. $G\varphi \to \varphi$ says: if φ will always be the case, then φ is the case now. But the future tense operator quantifies over *future* instants, not the present one. If R is the "strictly later than" relation, the current instant is not in its own future, so reflexivity fails and axiom T does not hold.

Tense logics are extensions of K. Necessitation applies: a logical truth is eternal, so it has always been true and always will be. The distribution axiom also holds for both G and H:

$$G(\varphi \to \psi) \to (G\varphi \to G\psi) \qquad\qquad H(\varphi \to \psi) \to (H\varphi \to H\psi)$$

Interaction principles. The future and past operators interact through the following axioms:

$$(GP) \qquad \varphi \to GP\varphi \quad \text{"If } \varphi \text{ is true now, it will always have been true."} \tag{1}$$

$$(HF) \qquad \varphi \to HF\varphi \quad \text{"If } \varphi \text{ is true now, it always was going to be true."} \tag{2}$$

11 Logics for Belief and Knowledge

Belief operators. We introduce a family of modal operators B_i for each agent i: $B_i\varphi$ is read "agent i believes that φ." Each B_i satisfies the rules of K; in particular, the necessitation rule applies.

Logical omniscience. Since necessitation holds, if φ is a logical truth, then $B_i\varphi$ for every agent i: every agent believes every logical truth. This is the **principle of logical omniscience**. It is descriptively unrealistic (human agents are not logically omniscient), but may be defensible as a normative idealisation (a rational agent *should* believe logical truths).

11.1 Axiom D for Beliefs

$$B_i\varphi \to \neg B_i\neg\varphi$$

If an agent accepts a proposition, the agent does not also accept its negation. This is a *consistency* requirement on beliefs. It corresponds to seriality: for every doxastic state, there is at least one world compatible with the agent's beliefs.

11.2 Axiom T/M and Epistemic Logic

$$B_i\varphi \to \varphi \qquad ???$$

This principle is *not* appropriate for belief: as Plato famously remarked in the *Gorgias*, beliefs can be false. But it does hold for **knowledge**, because knowledge is a *factive* concept: if an agent knows that φ, then φ is true. Writing K_i for "agent i knows that":

$$K_i\varphi \to \varphi$$

11.3 Positive Introspection

$$K_i\varphi \to K_iK_i\varphi \qquad\qquad B_i\varphi \to B_iB_i\varphi$$

If an agent knows (or believes) something, the agent knows (or believes) that they know (or believe) it. This corresponds to axiom 4 (transitivity).

11.4 Negative Introspection

$$\neg K_i\varphi \to K_i\neg K_i\varphi \qquad\qquad \neg B_i\varphi \to B_i\neg B_i\varphi$$

If an agent does not know (or believe) something, the agent knows (or believes) that they do not. This corresponds to axiom 5 (euclideanness). Adding both positive and negative introspection to epistemic logic yields the system $S5$ for knowledge—the standard framework for epistemic logic in game theory, computer science, and formal epistemology.

Part IV
Summary

12 What We Have Done

1. We introduced a **labelled natural deduction** system for K, where every formula carries a world label and accessibility assertions are explicit. The four modal rules ($\Box$E, $\Box$I, $\Diamond$I, $\Diamond$E) are structurally parallel to the quantifier rules in first-order logic.

2. We derived the **distribution axiom** $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and explained the **necessitation rule** from both proof-theoretic and model-theoretic perspectives.

3. We surveyed the principal **extensions of** K: the alethic system T (reflexivity), $S4$ (transitivity), B (symmetry), $S5$ (equivalence), the deontic system D (seriality), tense logics, and epistemic logics for belief and knowledge.