Modal Logic: Tableaux and Counter-Models

Solutions

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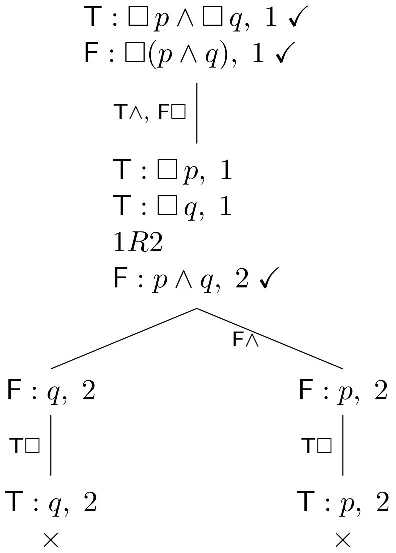
Contents

1 Exercise 1: Validity in K

1.1 (a) $(\Box p \wedge \Box q) \models_K \Box(p \wedge q)$

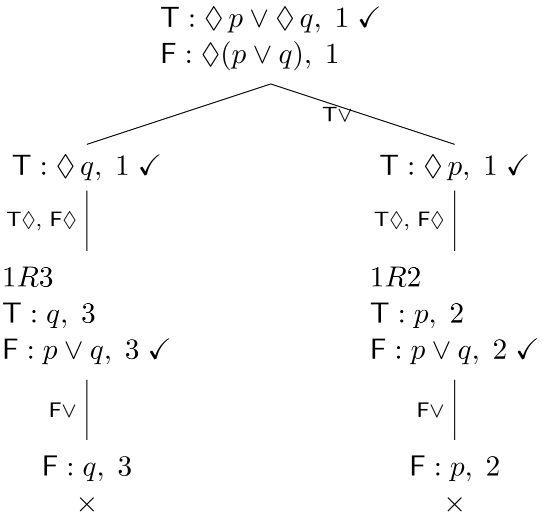


Explanation:

1. T∧: $\mathsf{T} : \Box p$, 1 and $\mathsf{T} : \Box q$, 1.
2. F□ (existential): introduce 2, $1R2$, $\mathsf{F} : p \wedge q$, 2.
3. F∧ (branching): $\mathsf{F} : p$, 2 or $\mathsf{F} : q$, 2.
4. Left: T□ on $\mathsf{T} : \Box p$ with $1R2$: $\mathsf{T} : p$, 2. Closes. Right: T□ on $\mathsf{T} : \Box q$ with $1R2$: $\mathsf{T} : q$, 2. Closes.

All branches close. The form is **valid**. □

1.2 (b) $\Diamond p \vee \Diamond q \models_K \Diamond(p \vee q)$



Explanation:

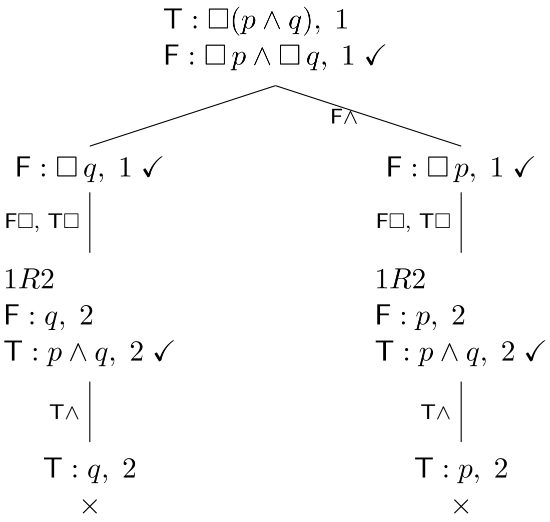
1. T∨ (branching): $\mathsf{T} : \Diamond p$, 1 or $\mathsf{T} : \Diamond q$, 1.

2. Left: $\mathsf{T}\Diamond$ (existential): $1R2$, $\mathsf{T} : p$, 2. Then $\mathsf{F}\Diamond$ (universal) with $1R2$: $\mathsf{F} : p \vee q$, 2. Then $\mathsf{F}\vee$: $\mathsf{F} : p$, 2 and $\mathsf{F} : q$, 2. Closes with $\mathsf{T} : p$, 2.

3. Right: symmetrically, $\mathsf{T} : q$, 3 and $\mathsf{F} : q$, 3. Closes.

Valid. □

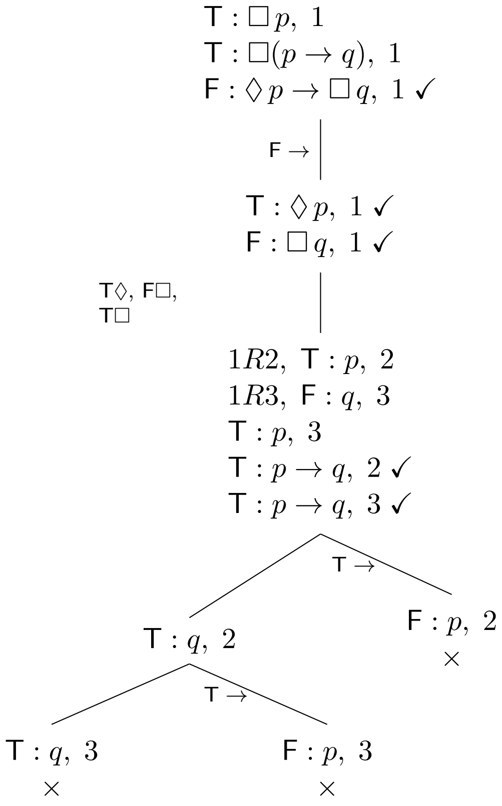
1.3 (c) $\Box(p \wedge q) \models_K \Box p \wedge \Box q$



Together with (a), this shows that $\Box(p \wedge q)$ and $\Box p \wedge \Box q$ are logically equivalent in K: necessity distributes over conjunction in both directions.

Valid. □

1.4 (d) $\Box p,\ \Box(p \to q) \models_K \Diamond p \to \Box q$



Explanation:

1. $\mathsf{F}\rightarrow$: $\mathsf{T} : \Diamond p$, 1 and $\mathsf{F} : \Box q$, 1.
2. Existential rules: $\mathsf{T}\Diamond$ gives $1R2$, $\mathsf{T} : p$, 2. Then $\mathsf{F}\Box$ gives $1R3$, $\mathsf{F} : q$, 3.
3. Universal rules with $1R2$ and $1R3$: propagate $\mathsf{T} : p$ and $\mathsf{T} : p \rightarrow q$ to both indices.
4. $\mathsf{T}\rightarrow$ at index 2: $\mathsf{F} : p$, 2 (closes) or $\mathsf{T} : q$, 2.
5. On surviving branch, $\mathsf{T}\rightarrow$ at index 3: $\mathsf{F} : p$, 3 (closes with $\mathsf{T} : p$, 3) or $\mathsf{T} : q$, 3 (closes with $\mathsf{F} : q$, 3).

Valid. $\square$

Remark 1.1. In fact, $\Box p$ and $\Box(p \rightarrow q)$ already entail $\Box q$ by the K-axiom, so $\Diamond p \rightarrow \Box q$ holds a fortiori (its consequent is true).

1.5 (e) $\Diamond p \wedge \Diamond q \models_K \Diamond(p \vee r)$

$\mathsf{T} : \Diamond p \wedge \Diamond q,\ 1$ ✓
$\mathsf{F} : \Diamond(p \vee r),\ 1$

$\mathsf{T}\wedge$, $\mathsf{T}\Diamond$, $\mathsf{F}\Diamond$

$\mathsf{T} : \Diamond p,\ 1$ ✓
$\mathsf{T} : \Diamond q,\ 1$ ✓
$1R2,\ \mathsf{T} : p,\ 2$
$1R3,\ \mathsf{T} : q,\ 3$
$\mathsf{F} : p \vee r,\ 2$ ✓
$\mathsf{F} : p \vee r,\ 3$

$\mathsf{F}\vee$

$\mathsf{F} : p,\ 2$
$\times$

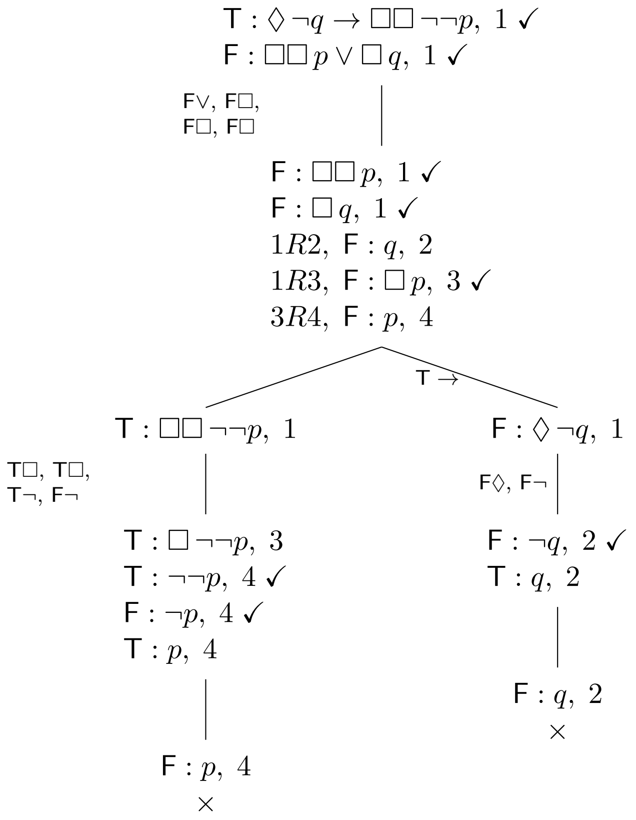
Explanation:

1. $\mathsf{T}\wedge$: $\mathsf{T} : \Diamond p$, 1 and $\mathsf{T} : \Diamond q$, 1.
2. $\mathsf{T}\Diamond$ (existential) on each: $1R2$, $\mathsf{T} : p$, 2 and $1R3$, $\mathsf{T} : q$, 3.
3. $\mathsf{F}\Diamond$ (universal) with $1R2$: $\mathsf{F} : p \vee r$, 2.
4. $\mathsf{F}\vee$: $\mathsf{F} : p$, 2 and $\mathsf{F} : r$, 2. Closes with $\mathsf{T} : p$, 2.

The branch already closes at index 2, without needing to process index 3: the witness for $\Diamond p$ provides a world where p is true, but $\mathsf{F} : \Diamond(p \vee r)$ forces $p \vee r$ to be false there.

Valid. $\square$

1.6 (f) $\Diamond \neg q \to \Box\Box \neg\neg p \models_K \Box\Box p \lor \Box q$



Explanation:

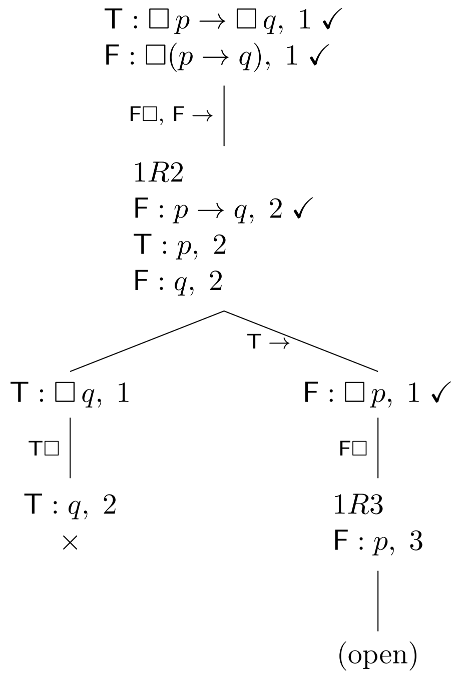
1. F∨: $\mathsf{F} : \Box\Box p, 1$ and $\mathsf{F} : \Box q, 1$.
2. Existential chain: F□ on $\mathsf{F} : \Box q$ gives $1R2$, $\mathsf{F} : q, 2$. Then F□ on $\mathsf{F} : \Box\Box p$ gives $1R3$, $\mathsf{F} : \Box p, 3$. Then F□ on $\mathsf{F} : \Box p, 3$ gives $3R4$, $\mathsf{F} : p, 4$.
3. T → on the premise (branching): $\mathsf{F} : \Diamond \neg q, 1$ or $\mathsf{T} : \Box\Box \neg\neg p, 1$.
4. Left branch: F◇ with $1R2$: $\mathsf{F} : \neg q, 2$. Then F¬: $\mathsf{T} : q, 2$. Closes with $\mathsf{F} : q, 2$.
5. Right branch: T□ with $1R3$: $\mathsf{T} : \Box \neg\neg p, 3$. Then T□ with $3R4$: $\mathsf{T} : \neg\neg p, 4$. Double negation elimination: $\mathsf{T} : p, 4$. Closes with $\mathsf{F} : p, 4$.

Valid. $\square$

2 Exercise 2: Counter-Models

2.1 (a) $\Box p \to \Box q \not\models_K \Box(p \to q)$

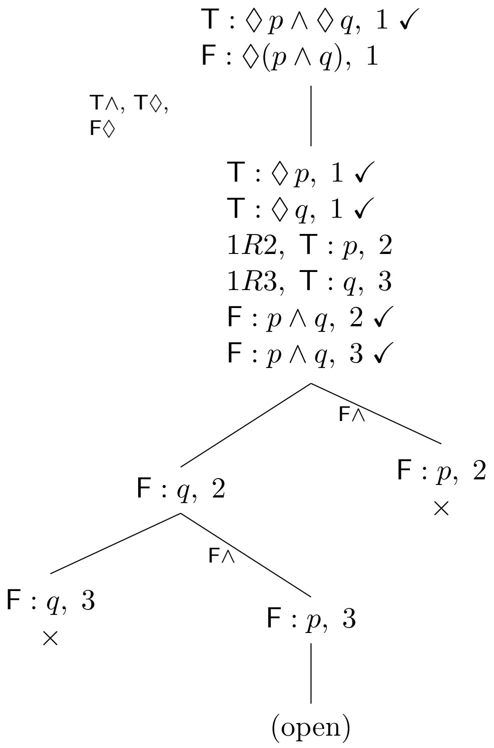
This is the example treated in the lecture notes (Section 7.1). We reproduce it here for completeness.



Counter-model. $W = \{1, 2, 3\}$, $R = \{\langle 1, 2\rangle, \langle 1, 3\rangle\}$, $I(p, 2) = \mathsf{T}$, $I(q, 2) = \mathsf{F}$, $I(p, 3) = \mathsf{F}$.

Verification. $\Box p$ is false at 1 (since p is false at 3), so $\Box p \to \Box q$ is vacuously true. But $p \to q$ is false at 2 (since p is true and q is false), so $\Box(p \to q)$ is false at 1. $\Box$

2.2 (b) $\Diamond p \wedge \Diamond q \not\models_K \Diamond(p \wedge q)$

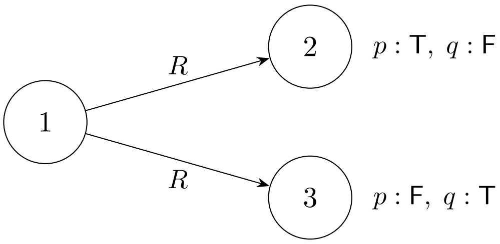


Explanation:

1. $\mathsf{T}\wedge$, then $\mathsf{T}\Diamond$ twice: $1R2$, $\mathsf{T} : p$, 2 and $1R3$, $\mathsf{T} : q$, 3.

2. $\mathsf{F}\Diamond$ (universal) with $1R2$ and $1R3$: $\mathsf{F} : p \wedge q,\ 2$ and $\mathsf{F} : p \wedge q,\ 3$.
3. $\mathsf{F}\wedge$ at index 2 (branching): $\mathsf{F} : p,\ 2$ (closes) or $\mathsf{F} : q,\ 2$.
4. On surviving branch, $\mathsf{F}\wedge$ at index 3: $\mathsf{F} : p,\ 3$ (**open**) or $\mathsf{F} : q,\ 3$ (closes).

Counter-model. $W = \{1, 2, 3\}$, $R = \{\langle 1, 2\rangle, \langle 1, 3\rangle\}$, $I(p, 2) = \mathsf{T}$, $I(q, 2) = \mathsf{F}$, $I(p, 3) = \mathsf{F}$, $I(q, 3) = \mathsf{T}$.



Verification. $\Diamond p$ is true at 1 (witness: index 2); $\Diamond q$ is true at 1 (witness: index 3). But $p \wedge q$ is false at both accessible indices, so $\Diamond(p \wedge q)$ is false at 1.

Remark 2.1. The witnesses for $\Diamond p$ and $\Diamond q$ are at *different* worlds: p and q are each possible, but they are never true *together*. Possibility does not distribute over conjunction.

2.3 (c) $\neg\Diamond\neg p \not\models_K \Diamond p$

$\mathsf{T} : \neg\Diamond\neg p,\ 1\ \checkmark$
$\mathsf{F} : \Diamond p,\ 1$

$\mathsf{T}\neg$

$\mathsf{F} : \Diamond\neg p,\ 1$

(open)

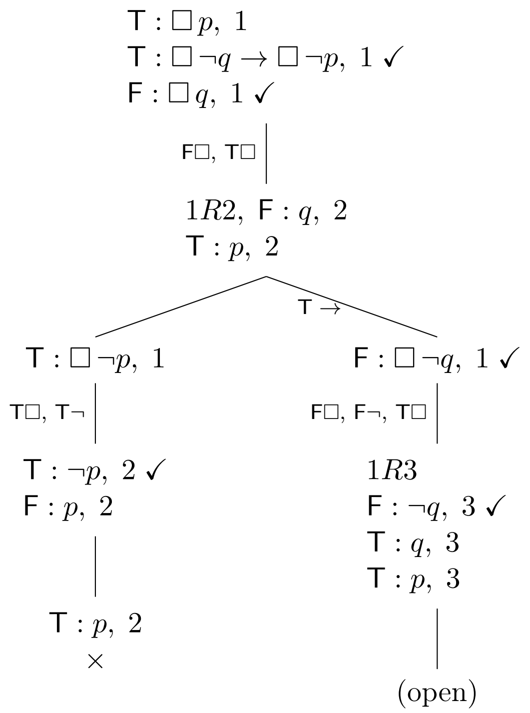
Explanation: $\mathsf{T}\neg$ gives $\mathsf{F} : \Diamond\neg p,\ 1$. Both $\mathsf{F} : \Diamond p$ and $\mathsf{F} : \Diamond\neg p$ are universal rules, but no accessibility pair $1Rj$ exists on the branch, so neither can fire. The branch is open.

Counter-model. $W = \{1\}$, $R = \varnothing$, $I(p, 1)$ arbitrary.

Verification. $\neg\Diamond\neg p$ is equivalent to $\Box p$. With $R = \varnothing$, $\Box p$ is vacuously true and $\Diamond p$ is false. This is the familiar dead-end world phenomenon: $\Box p \not\models_K \Diamond p$.

$\Box$

2.4 (d) $\Box p,\ \Box\neg q \to \Box\neg p \not\models_K \Box q$



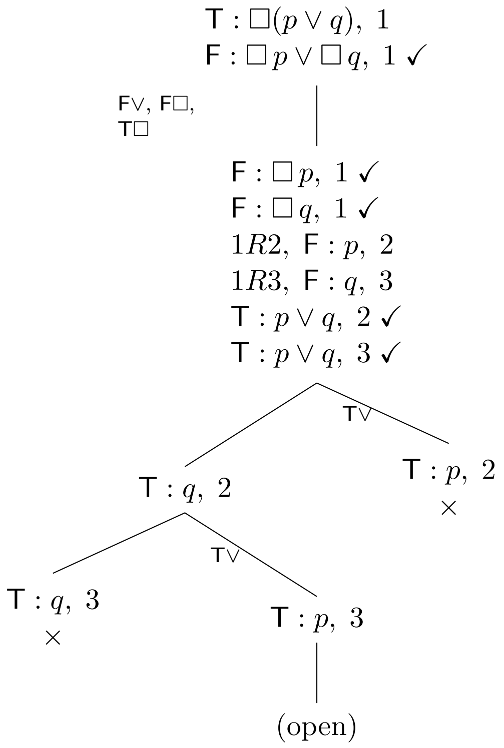
Counter-model. $W = \{1, 2, 3\}$, $R = \{\langle 1, 2\rangle, \langle 1, 3\rangle\}$, $I(p, 2) = \mathsf{T}$, $I(q, 2) = \mathsf{F}$, $I(p, 3) = \mathsf{T}$, $I(q, 3) = \mathsf{T}$.

Verification. $\Box p$ is true at 1 (p true at both 2 and 3). $\Box\neg q$ is false at 1 ($\neg q$ fails at 3), so $\Box\neg q \to \Box\neg p$ is vacuously true. But $\Box q$ is false (q fails at 2). □

3 Exercise 3: Valid or Invalid?

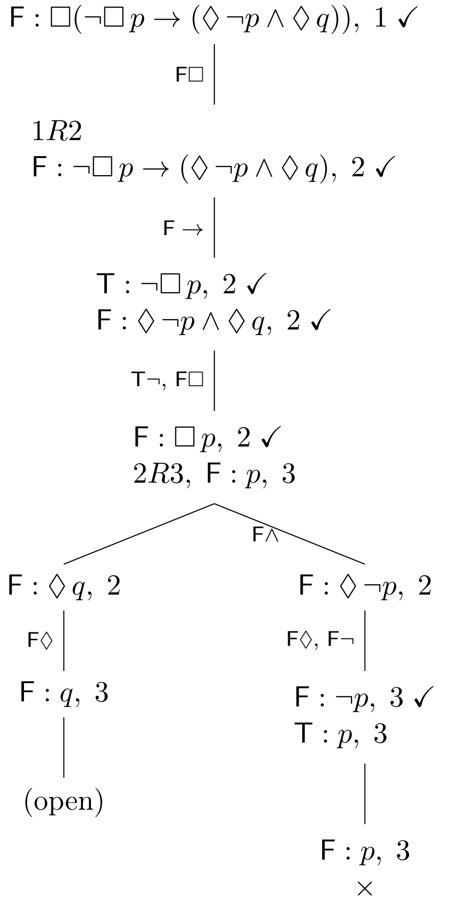
3.1 (a) $\Box(p \lor q) \not\models_K \Box p \lor \Box q$ — Invalid

This is the counter-model for distribution over disjunction treated in the lecture notes (Section 7.2).



Counter-model. $W = \{1,2,3\}$, $R = \{\langle 1,2\rangle, \langle 1,3\rangle\}$, $I(p,2) = \mathsf{F}$, $I(q,2) = \mathsf{T}$, $I(p,3) = \mathsf{T}$, $I(q,3) = \mathsf{F}$. Each accessible world makes $p \vee q$ true by choosing a *different* disjunct. $\Box$

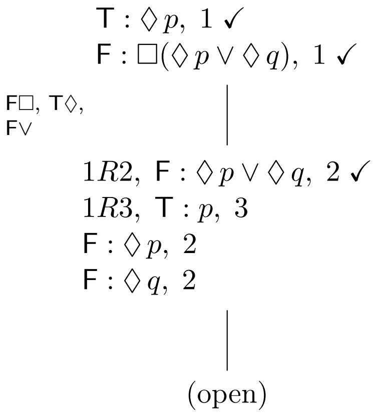
3.2 (b) $\not\models_K \Box(\neg\Box p \to (\Diamond\neg p \wedge \Diamond q))$ — Invalid



Counter-model. $W = \{1, 2, 3\}$, $R = \{\langle 1, 2\rangle, \langle 2, 3\rangle\}$, $I(p, 3) = \mathsf{F}$, $I(q, 3) = \mathsf{F}$.

Verification. At 3: p is false. At 2: $\Box p$ is false (since $2R3$ and p false at 3), so $\neg\Box p$ is true. $\Diamond\neg p$ is true ($\neg p$ at 3). $\Diamond q$ is false (q false at 3, the only world accessible from 2). So $\Diamond\neg p \wedge \Diamond q$ is false. The conditional $\neg\Box p \to (\Diamond\neg p \wedge \Diamond q)$ is false at 2. Hence the boxed formula is false at 1. $\Box$

3.3 (c) $\Diamond p \not\models_K \Box(\Diamond p \vee \Diamond q)$ — Invalid

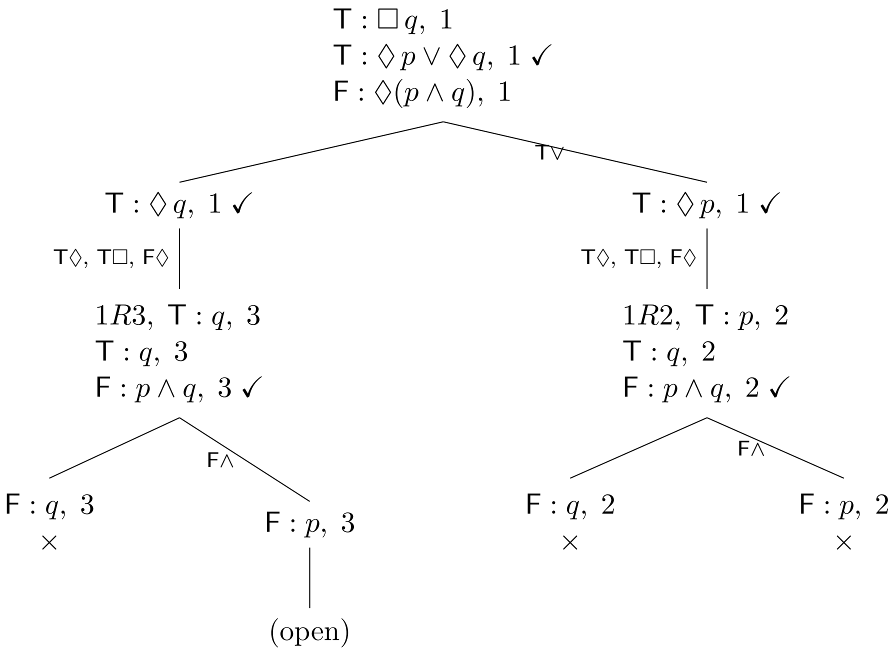


No accessibility pair $2Rj$ exists, so neither $\mathsf{F} : \Diamond p$, 2 nor $\mathsf{F} : \Diamond q$, 2 can fire.

Counter-model. $W = \{1, 2, 3\}$, $R = \{\langle 1, 2\rangle, \langle 1, 3\rangle\}$, $I(p, 3) = \mathsf{T}$.

Verification. $\Diamond p$ true at 1 (witness: 3). At 2: no world is accessible from 2, so $\Diamond p$ and $\Diamond q$ are both false, hence $\Diamond p \vee \Diamond q$ false at 2. Thus $\Box(\Diamond p \vee \Diamond q)$ false at 1. $\Box$

3.4 (d) $\Box q,\ \Diamond p \vee \Diamond q \not\models_K \Diamond(p \wedge q)$ — Invalid



The left main branch closes (both sub-branches), but the right main branch has an open sub-branch.

Counter-model. $W = \{1, 3\}$, $R = \{\langle 1, 3\rangle\}$, $I(q, 3) = \mathsf{T}$, $I(p, 3) = \mathsf{F}$.

Verification. $\Box q$ true at 1 (q true at the only accessible world). $\Diamond q$ true at 1, so $\Diamond p \vee \Diamond q$ true. But $p \wedge q$ false at 3 (p false), so $\Diamond(p \wedge q)$ false at 1. $\Box$

3.5 (e) $\Box p \not\models_K \Diamond p$ — Invalid

Already treated in the lecture notes (Section 9.3).

Counter-model. $W = \{1\}$, $R = \varnothing$. $\Box p$ is vacuously true; $\Diamond p$ is false (no accessible world). This is the dead-end world phenomenon. Seriality of R (system D) would block this. $\Box$

3.6 (f) $\Box p \not\models_K \Box\Box p$ — Invalid

$\mathsf{T} : \Box p,\ 1$
$\mathsf{F} : \Box\Box p,\ 1\ \checkmark$

F□, T□

$1R2,\ \mathsf{F} : \Box p,\ 2\ \checkmark$
$\mathsf{T} : p,\ 2$

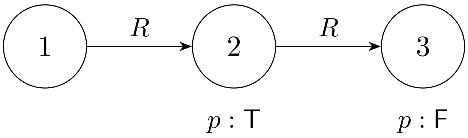
F□

$2R3,\ \mathsf{F} : p,\ 3$

(open)

Note that T□ on $\mathsf{T} : \Box p, 1$ only applies to pairs $1Rj$, *not* to $2R3$. So $\mathsf{T} : p$ cannot be propagated to index 3.

Counter-model. $W = \{1, 2, 3\}$, $R = \{\langle 1, 2\rangle, \langle 2, 3\rangle\}$, $I(p, 2) = \mathsf{T}$, $I(p, 3) = \mathsf{F}$.



Verification. $\Box p$ true at 1 (p true at 2, the only world 1 sees). $\Box p$ false at 2 (p false at 3). So $\Box\Box p$ false at 1.

Remark 3.1. The schema $\Box p \to \Box\Box p$ is axiom 4. It corresponds to transitivity of R: the counter-model has $1R2$ and $2R3$ but not $1R3$. Adding transitivity yields the logic $S4$.

3.7 (g) $p \not\models_K \Box\Diamond p$ — Invalid

$\mathsf{T} : p,\ 1$
$\mathsf{F} : \Box\Diamond p,\ 1\ \checkmark$

F□

$1R2$
$\mathsf{F} : \Diamond p,\ 2$

(open)

$\mathsf{F} : \Diamond p, 2$ is universal but no $2Rj$ exists, so it cannot fire.

Counter-model. $W = \{1, 2\}$, $R = \{\langle 1, 2\rangle\}$, $I(p, 1) = \mathsf{T}$.

Verification. p true at 1. At 2: no accessible world, so $\Diamond p$ false. Hence $\Box\Diamond p$ false at 1. □

3.8 (h) $\Diamond p \not\models_K \Box\Diamond p$ — Invalid

$\mathsf{T} : \Diamond p,\ 1$ ✓
$\mathsf{F} : \Box\Diamond p,\ 1$ ✓

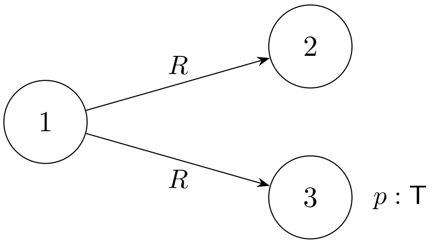
$\mathsf{F}\Box$, $\mathsf{T}\Diamond$

$1R2,\ \mathsf{F} : \Diamond p,\ 2$
$1R3,\ \mathsf{T} : p,\ 3$

(open)

$\mathsf{F} : \Diamond p$, 2 requires $\mathsf{F} : p$, j for all j with $2Rj$. No such j exists. No contradiction.

Counter-model. $W = \{1, 2, 3\}$, $R = \{\langle 1, 2\rangle, \langle 1, 3\rangle\}$, $I(p, 3) = \mathsf{T}$.



Verification. $\Diamond p$ true at 1 (witness: 3). At 2: 2 sees nothing, so $\Diamond p$ false at 2. Hence $\Box\Diamond p$ false at 1.

Remark 3.2. The schema $\Diamond p \to \Box\Diamond p$ is axiom 5. It corresponds to the euclideanness of R (if wRu and wRv then uRv): the counter-model has $1R2$ and $1R3$ but not $2R3$. Adding euclideanness (together with reflexivity and transitivity) yields $S5$.