Modal Logic I

Semantics of K

Introduction

Non-classical logics

- Revise classical logic by rejecting certain laws : non-bivalent logics, intuitionistic logic, relevant logic.
- Extend classical logic by formalizing new logical concepts :
  - While conserving all classical laws: second-order logic.
  - While changing some laws: **modal logics.**

Extension vs Intension

- In FOL, the extension of an expression is either its reference, in the case of a proper name, or the set of objects to which the expression applies.
- Two expressions with the same extension may differ in meaning; in this case, they are said to have different **intensions**.
    - Exemple : $\{x \mid \text{x has a heart}\} = \{x \mid \text{x has a kidney}\}$, nevertheless, « having a heart » and « having a kidney » have distinct meanings, hence different intensions.

Extensionality

- In classical logic, the truth value of a statement depends only on (i) the interpretation of the symbols in the statement and (ii) the syntactic structure of the statement.
- Interpretation is reduced to reference or extension: a domain individual for an individual constant, a set for a predicate, a function for a function symbol.
- Therefore, the truth value of a statement only depends on the reference or extension of the symbols in the statement, and on the syntactic structure of the statement.
- Consequence: expressions with the same extensions or references can be substituted to each other without changing the truth value of statements (= *salva veritate*).

Illustrations

- Co-referent proper names : Razib believes Eric Rohmer directed La boulangère de Monceau ==> Razib believes Maurice Schérer directed La boulangère de Monceau.
- Co-extensive predicates : Lila knows that if someone has a heart, they have a heart ==> Lili knows that if someone has a heart, they have a kidney.
- Co-denoting statements : It is necessary that 2+2=4 ==> It is necessary that M. Ludwig has a stylish bike.

Core idea for intensional semantics

- **Abandon extensionality**
- The truth values of formulas, as well as the denotations of sub-sentential expressions, are **relativized to evaluation circumstances**.
- So, a statement like « M. Ludwig has a stylish bike » can be considered as true in some situations, and false in others; while « 2+2=4 » would be always considered true.
- Such evaluation circumstances are called **indices**.

Syntax and vocabulary

- We add a new primitive symbol to LP: $\Box$
- Another new symbol can be defined from $\Box$: $\Diamond =_{Def} \neg \Box \neg$
- We also add a new rule to LP's grammar:
    - If ϕ is a well-formed formula, $\Box \phi$ also is a well-formed formula.
- Exercise: show that if ϕ is a wff, so is $\Diamond \phi$

Intensional models

- **Definition** : an intensional model is a pair $< W, I >$ where W is a non-empty set of indices, and I an interpretation function with two argument places— one place for propositional variables, and another one for indices.
    - I gives a truth-value for each pair (propositional variable, index).
- As an illustration let us consider a model such that $W = \{\omega_1, \omega_2\}, I(p, \omega_1) = V$ et $I(p, \omega_2) = F$
- We could consider p as translating « M. Ludwig's bike is stylish »; in the model, the statement is true relative to ω_1, and false relative to ω_2.

My bike in the first index

My bike in the second one

Notations

- $M, \omega_1 \vDash p$ (or: $M \vDash_{\omega_1} p$)
- $M, \omega_2 \nvDash p$

Tableaux for intensional semantics

- We just **relativize nodes to indices in our tableaux**. For convenience, we use numerals to represent the indices
- For instance, let us suppose a model such that $V : p \wedge q$ relative to an index, but $F : q$ in this very index.
- Here is the tree we get :
-

$V : p \wedge q, 1$

$F : p, 1$

$V : p, 1$

$V : q, 1$

Truth-conditions for modal sentences

- The core idea of intensional semantics is important, but it is not sufficient.
- As an illustration, let us consider the deontic interpretation of necessity, so let's read $\Box\phi$ as meaning « it is obligatory that ϕ ».
- If we rest on Leibniz's intuition that necessity should be equated to « truth in all possible worlds », we would get :
  - $\models \Box\phi \rightarrow \phi$
- If ϕ is true in all possible world, hence in any index, it is true in the index representing the real world.
- But this is incorrect for the deontic interpretation of $\Box$: we all know that in the real world, what is obligatory can very well not be true!
- So in our truth-definition, **we should not consider all indices, but only the relevant ones** — for instance as far as the deontic interpretation is concerned, the ones in which the deontic norms are satisfied.

The idea of an accessibility relation

- How are we going to determine which indices are relevant when defining truth?
- **Saul Kripke, core idea of modal logics**: the set of relevant indices in order to verify whether $\Box\phi$ is true or not in index ω are all the indices that are accessible from ω.
  - We introduce an accessibility (binary) relation in the meta-language in order to express this :
- **Definition** : a **Kripke model** is a triplet $< W, R, I >$ where W is a set of indices, R is an accessibility relation defined on $W \times W$, and I is a two-argument places interpretation function that gives the truth-value of each propositional variable in each index.
  - The pair $< W, R >$ is called a **Kripke frame**.

Truth-definition for K

- $M, w \vDash \Box \phi$ iff for all w' such that Rww', $M, w' \vDash \phi$
- $M, w \vDash \diamond \phi$ iff there is a w' such that Rww', and $M, w' \vDash \phi$
- We keep the usual truth-tables for the propositional concepts:
- $M, w \vDash \neg\phi$ ssi $M, w \nvDash \phi$
- $M, w \vDash \phi \wedge \psi$ ssi $M, w \vDash \phi$ et $M, w \vDash \psi$
- $M, w \vDash \phi \vee \psi$ ssi $M, w \vDash \phi$ ou $M, w \vDash \psi$
- $M, w \vDash \phi \rightarrow \psi$ ssi $M, w \nvDash \phi$ ou $M, w \vDash \psi$
- $M, w \nvDash \bot$ (for any model and any index of this model).

Important semantic concepts

- A formula ϕ **K-satisfiable** iff there is a model $< W, R, I >$ such that there is an index $w \in W$ such that $M, w \vDash \phi$.
- **A set Γ of formulas is K-satisfiable** off there is a model $< W, R, I >$ such that there is an index $w \in W$ such that : for any $\gamma \in \Gamma$ $M, w \vDash \gamma$
- An argument form F with premisses $\phi_1 \ldots \phi_n$ and conclusion ψ has a **counter-model** iff the set $\{\phi_1 \ldots \phi_n, \neg\psi\}$ is satisfiable.
- **An argument form is K-valid iff it has no counter-model.**
- **A formula ϕ is a logical truth in K iff it is true : (i) in any Kripke model M (ii) relative to each index in M.**
  - **Notation :** $\vDash_K \phi$

Tableaux rules for $\Box$

$$
\begin{array}{c}
wRw' \\
T : \Box \varphi, w \\
| \\
T : \varphi, w'
\end{array}
\qquad\qquad
\begin{array}{c}
F : \Box \varphi, w \\
| \\
wRw' \\
| \\
F : \varphi, w'
\end{array}
$$

Explanations

- The rule for $T : \Box \phi, \omega$ is just an application of the truth-definition for $\Box \phi$: if $\Box \phi$ is true in the index of a model, then ϕ is true in all the indices accessible from it.

- The rule for $F : \Box \phi$ stems from the fact that if $\Box \phi$ is false in an index in a model, then there is another index accessible from it in which ϕ is false.

- **Important** : please note that the rule for $F : \Box \phi$ has an **existential import**; informally, we can see it as **introducing a new index in a tableau**; in contrast with that, the rule for $T : \Box \phi, \omega$ has an universal important and does not introduce new indices.

Tableaux rules for $\Diamond$

$$T : \Diamond \varphi, w$$

$$|$$

$$wRw'$$

$$|$$

$$T : \varphi, w'$$

$$wRw'$$

$$F : \Diamond \varphi, w$$

$$|$$

$$F : \varphi, w'$$

How to apply the rules for tableaux?

- Apply the rules for Propositional logic until no more rules can be applied.
- Then apply the rules with an existential import, that is: $T : \diamond \phi$ and $F : \Box \phi$; introduce a new index for each such rule
  - Go to step one.
- Then apply the rules with a universal import.
  - Go to step one.
- A tree finished iff all its branches are closed, or all the relevant rules have been applied.

Core theorem for K: $\vDash_K \Box(p \to q) \to (\Box p \to \Box q)$

$F : \Box(p \to q) \to (\Box p \to \Box q), 1$

$T : \Box(p \to q), 1$

$F : \Box p \to \Box q, 1$

$T : \Box p, 1$

$F : \Box q, 1$

$1 R 2$

$F : q, 2$

$T : p, 2$

$T : p \to q, 2$

$F : p, 2$	$T : q, 2$
X	X

$$\models_K \Box(p \to q) \to (\Diamond p \to \Diamond q)$$

$\Box(p \to q)$

$\neg(\Diamond p \to \Diamond q)$

$\Diamond p$

$\neg \Diamond q$

$\downarrow$

p

$p \to q$

q

$\neg q$

$\times$

V: $\Box(p \to q)$, 1

F: $\Diamond p \to \Diamond q$, 1

V: $\Diamond p$, 1

F: $\Diamond q$, 1

1R2

V: p, 2

V: $p \to q$, 2

F: q, 2

F: p, 2 $\times$ | V: q, 2 $\times$

$$\Box(p \to q) \land \Box(q \to r) \vdash \Box(p \to r)$$

V: $\Box(p \to q) \land \Box(q \to r)$, 1

F: $\Box(p \to r)$, 1

V: $\Box(p \to q)$, 1

V: $\Box(q \to r)$, 1

1 R 2

F: $p \to r$, 2

V: $p \to q$, 2

V: $q \to r$, 2

V: p, 2

F: r, 2

F, p, 2 ×

V: q, 2

F: q, 2 ×

V: r, 2 ×

Building counter-models with tableaux

- If a finished tableau has one (or more) open branch, we can use this branch to build a counter-model
  - The set of indices induced by the branch contains all the indices that occur on it
  - We define the interpretation function by stipulating that all the equations on the branch are to be verified on the interpretation.
- As a first exemple, let us consider :
  - $\nvDash_K (\Box p \rightarrow \Box q) \rightarrow \Box (p \rightarrow q)$

$T: \Box p \to \Box q, 1$

$F: \Box(p \to q), 1$

$F: \Box p, 1$

$1R2$

$F: p, 2$

$1R3$

$F: p \to q, 3$

$T: p, 3$

$F: q, 3$

$T: \Box q, 1$

$1R4$

$F: p \to q, 4$

$T: p, 4$

$F: q, 4$

$T: q, 4$

$\times$

- The left branch can be used to build a counter-model :
  - $W = \{1,2,3\}; R = \{ < 1,2 > , < 1,3 > \}$
  - $I(p,2) = F; I(p,3) = T; I(q,3) = F$
- $M \nvDash_2 p$, and $1R2$, so $M \nvDash_1 \Box p$. So the conditional $\Box p \to \Box q$ is true in 1.
- But $1R3$, and $p \to q$ is false in 3. So $M \nvDash_1 \Box (p \to q)$
- We can conclude that $M \nvDash_1 (\Box p \to \Box q) \to \Box (p \to q)$, and therefore $\nvDash_K (\Box p \to \Box q) \to \Box (p \to q)$

Conclusion: some laws that K does not validate

- K is very general and inclusive: it is supposed to apply to any modal concept.
- This is why is does not validate some particular modal laws, that are true in some interpretations of $\Box$ but not in all interpretations.
- For instance, let us consider : $\nvDash_K \Box\phi \rightarrow \phi$ and $\nvDash_K \phi \rightarrow \Diamond\phi$
- Intuitively, we know that these laws should not be validated by K, because they are false in the deontic interpretation of $\Box$.
- Let us check that with tableaux:

$F: \Box p \to p, 1$

$V: \Box p, 1$

$F: p, 1$

Counter-model: $M = \langle W, R, \mathcal{V} \rangle$

$W = \{1\}$ $\mathcal{V}(p, 1) = F$

$R = \emptyset$

$M \models_1 \Box p$

$M \not\models_1 p$

$F: p \to \Diamond p, 1$

$T: p, 1$

$F: \Diamond p, 1$

counter-model $M = \langle W, R, I \rangle$

$R = \emptyset$ $I(p,1) = T$

$W = \{1\}$

$M \not\models_1 \Diamond p$ $M \models_1 p$

$$\Box p \to \Diamond p$$

F: $\Box p \to \Diamond p$, 1

T: $\Box p$, 1

F: $\Diamond p$, 1

Counter. model: $\mathcal{M} = \langle W, R, V \rangle$

$W = \{1\}$ $\quad$ $V(p) = T$ or F

$R = \emptyset$

$\mathcal{M} \models_1 \Box p$

$\mathcal{M} \not\models_1 \Diamond p$