Modal Logic I

Semantics of K

Pascal Ludwig

Contents

Part I

From Classical Logic to Modal Logic

1 Introduction: Non-Classical Logics

Classical propositional logic and first-order logic provide an extraordinarily powerful framework, but they rest on assumptions that are not always appropriate. There are broadly two strategies for departing from classical logic.

Revising classical logic. One can *reject* certain classical laws. Examples include many-valued logics, which drop bivalence; intuitionistic logic, which drops the law of excluded middle and double negation elimination; and relevant logics, which impose a relevance condition on the conditional.

Extending classical logic. Alternatively, one can *add* new expressive resources while retaining some or all classical laws. Second-order logic, for instance, adds quantification over properties and relations while conserving every first-order law. Modal logics add operators such as $\Box$ ("it is necessary that...") and $\Diamond$ ("it is possible that..."); in doing so they modify certain substitution principles, for reasons we shall now examine.

2 Extension, Intension, and Extensionality

2.1 Extension versus Intension

In first-order logic, every expression is assigned an *extension* relative to a structure:

- An individual constant is assigned an object in the domain (its *reference*).
- A predicate is assigned the set of objects to which it applies.
- A sentence is assigned a truth value.

Two expressions can share the same extension while differing in meaning. When this happens, we say they have distinct *intensions*.

Example 2.1. The predicates "has a heart" and "has a kidney" are co-extensive:

$$\{x \mid x \text{ has a heart}\} = \{x \mid x \text{ has a kidney}\}.$$

Yet the two expressions plainly differ in meaning; they contribute different information, and it is conceivable that they come apart. Their *intensions* are different.

2.2 The Extensionality of Classical Logic

In classical logic, the truth value of a statement is determined entirely by two factors:

1. the *extension* (reference) of its non-logical symbols, and
2. the *syntactic structure* of the statement.

No further information—no "meaning" beyond reference—plays any role. This yields a powerful substitution principle:

Extensionality Principle. Expressions with the same extensions or references can be substituted for one another *salva veritate*, that is, without changing the truth value of any statement in which they occur.

2.3 Failures of Extensionality

When we move to natural-language contexts involving belief, knowledge, or necessity, the extensionality principle visibly breaks down.

Co-referent proper names. "Eric Rohmer" and "Maurice Schérer" refer to the same individual. Yet from

> Razib believes Eric Rohmer directed *La boulangère de Monceau*

we cannot conclude

> Razib believes Maurice Schérer directed *La boulangère de Monceau*.

Razib may simply not know that Eric Rohmer *is* Maurice Schérer.

Co-extensive predicates. From "Lila knows that if someone has a heart, they have a heart" it does not follow that "Lila knows that if someone has a heart, they have a kidney."

Co-denoting statements. "$2 + 2 = 4$" and "M. Ludwig has a stylish bike" may well have the same truth value, yet "It is necessary that $2+2 = 4$" is true, while "It is necessary that M. Ludwig has a stylish bike" is not.

These examples show that contexts created by "believes", "knows", "it is necessary that", etc., are *intensional*: the truth value of the whole depends on more than the extensions of its parts.

3 The Core Idea of Intensional Semantics

How should one model intensionality? The strategy is to *abandon* the assumption that each expression has a single, fixed denotation, and instead **relativise** denotations and truth values to *evaluation circumstances*.

> **Key idea.** A statement such as "M. Ludwig has a stylish bike" can be true in some circumstances and false in others, whereas "$2 + 2 = 4$" is true in all of them. The evaluation circumstances are called **indices**.

At this stage, one need not say much about what indices *are*. In an alethic (truth/necessity) reading, they are naturally thought of as *possible worlds*; in a temporal reading, they could be instants of time; in a deontic reading, they could be idealised situations in which all obligations are fulfilled. The formal machinery is deliberately neutral.

4 Syntax of Modal Propositional Logic

We extend the language $\mathcal{L}_P$ of propositional logic by adding a single new primitive symbol:

$$\Box \quad \text{("box"—read: "it is necessary that").}$$

A second operator is defined from $\Box$:

$$\Diamond\,\varphi \;=_{\text{Def}}\; \neg\,\Box\,\neg\,\varphi \quad \text{("diamond"—read: "it is possible that").}$$

The grammar of $\mathcal{L}_P$ is supplemented by a single new clause:

> If φ is a well-formed formula, then $\Box\,\varphi$ is a well-formed formula.

Exercise 4.1. Show that if φ is a well-formed formula, so is $\Diamond\,\varphi$.

Solution. Suppose φ is a wff. By the rule for negation, $\neg\,\varphi$ is a wff. By the new rule, $\Box\,\neg\,\varphi$ is a wff. By the rule for negation again, $\neg\,\Box\,\neg\,\varphi$ is a wff. Since $\Diamond\,\varphi$ is defined as $\neg\,\Box\,\neg\,\varphi$, it follows that $\Diamond\,\varphi$ is a wff. $\Box$

5 Intensional Models (Without Accessibility)

Before introducing the full Kripke semantics, it is instructive to work with a simpler notion of model in which the set of indices carries no additional structure.

Definition 5.1 (Intensional model)**.** An **intensional model** is a pair $\langle W, I \rangle$, where:

- W is a non-empty set of **indices**, and
- I is an **interpretation function** that assigns a truth value to each pair (propositional variable, index).

Formally, $I\colon \mathrm{Var} \times W \to \{\mathsf{T}, \mathsf{F}\}$.

Example 5.1. Consider a model $\mathcal{M} = \langle W, I \rangle$ with $W = \{\omega_1, \omega_2\}$, $I(p, \omega_1) = \mathsf{T}$, and $I(p, \omega_2) = \mathsf{F}$.

One could read p as "M. Ludwig's bike is stylish". In this model, the statement is true at ω_1 and false at ω_2.

We write $\mathcal{M}, \omega \models p$ (or equivalently $\mathcal{M} \models_\omega p$) to mean that the propositional variable p is true in $\mathcal{M}$ at index ω, and $\mathcal{M}, \omega \not\models p$ to mean that it is false there.

5.1 Tableaux for Intensional Semantics

The tableau method carries over straightforwardly: one simply annotates each signed formula with the index at which it is being evaluated. For convenience, we use numerals $(1, 2, 3, \ldots)$ to represent indices.

Example 5.2. Suppose that at some index in a model we have $\mathsf{T}\colon p \wedge q$ and $\mathsf{F}\colon q$. We build the tableau at index 1:

$$
\begin{array}{c}
\mathsf{T} : p \wedge q,\ 1\ \checkmark \\
\mathsf{F} : q,\ 1 \\
| \\
\mathsf{T} : p,\ 1 \\
\mathsf{T} : q,\ 1 \\
\times
\end{array}
$$

The branch closes because it contains both $\mathsf{T} : q$, 1 and $\mathsf{F} : q$, 1.

Remark 5.1. At this stage, nothing connects different indices to one another. The propositional rules operate within a single index, exactly as in classical logic. The novelty—and the need for an accessibility relation—arises when we ask what $\Box\varphi$ means.

Part II
Kripke Semantics

6 Why an Accessibility Relation Is Needed

The simplest proposal for the truth-conditions of $\Box\varphi$ goes back to Leibniz:

> $\Box\varphi$ is true iff φ is true *in all indices.*

Under this definition, one immediately obtains the schema $\models \Box\varphi \to \varphi$: if φ is true in *all* indices, it is certainly true in whichever index we happen to be evaluating at.

Problem: the deontic interpretation. Consider the **deontic** interpretation of $\Box$, reading $\Box\varphi$ as "it is obligatory that φ". The schema $\Box\varphi \to \varphi$ would then say: whatever is obligatory is actual. But this is plainly incorrect—obligations can be violated.

More generally, different readings of $\Box$ validate different principles. A semantics that aims to be *general* should not build in assumptions that are appropriate only for some readings.

Solution: restrict the quantification. Instead of quantifying over *all* indices, we quantify only over the *relevant* ones—those that are **accessible** from the index of evaluation.

- Under the alethic reading, the indices accessible from w are the "metaphysically possible" worlds relative to w.
- Under the deontic reading, they are the worlds in which the norms operative at w are satisfied.
- Under the epistemic reading, they are the worlds compatible with what the agent knows at w.

Which indices count as accessible is encoded by a binary relation on the set of indices.

7 Kripke Models and the Truth Definition for K

7.1 Kripke Models

Definition 7.1 (Kripke model)**.** A **Kripke model** is a triple $\mathcal{M} = \langle W, R, I\rangle$, where:

- W is a non-empty set of **indices** (or *possible worlds*),
- $R \subseteq W \times W$ is a binary **accessibility relation** on W, and
- $I\colon \mathrm{Var} \times W \to \{\mathsf{T}, \mathsf{F}\}$ is an interpretation function.

The pair $\langle W, R\rangle$ is called a **Kripke frame**.

The accessibility relation R is left completely unconstrained in the minimal system K. Imposing structural conditions on R—reflexivity, transitivity, symmetry, etc.—yields stronger modal logics (T, $S4$, $S5$, ...), which we shall study in the sequel.

7.2 Truth Definition for K

Let $\mathcal{M} = \langle W, R, I\rangle$ be a Kripke model and $w \in W$.

Definition 7.2 (Satisfaction in K)**.** The relation $\mathcal{M}, w \models \varphi$ is defined recursively:

$$\mathcal{M}, w \models p \quad \text{iff } I(p, w) = \mathsf{T} \quad \text{(for a propositional variable } p\text{)}, \tag{1}$$

$$\mathcal{M}, w \models \neg\varphi \quad \text{iff } \mathcal{M}, w \not\models \varphi, \tag{2}$$

$$\mathcal{M}, w \models \varphi \wedge \psi \quad \text{iff } \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi, \tag{3}$$

$$\mathcal{M}, w \models \varphi \vee \psi \quad \text{iff } \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi, \tag{4}$$

$$\mathcal{M}, w \models \varphi \to \psi \quad \text{iff } \mathcal{M}, w \not\models \varphi \text{ or } \mathcal{M}, w \models \psi, \tag{5}$$

$$\mathcal{M}, w \not\models \bot, \tag{6}$$

$$\mathcal{M}, w \models \Box\varphi \quad \text{iff for all } w' \text{ such that } Rww',\ \mathcal{M}, w' \models \varphi, \tag{7}$$

$$\mathcal{M}, w \models \Diamond\varphi \quad \text{iff there exists } w' \text{ such that } Rww' \text{ and } \mathcal{M}, w' \models \varphi. \tag{8}$$

Remark 7.1. The clauses for the Boolean connectives (2)–(6) are exactly the classical truth-table conditions, relativised to an index. The only genuinely new clauses are (7) and (8).

Note that the clause for $\Diamond$ is derivable from those for $\neg$ and $\Box$: $\mathcal{M}, w \models \Diamond\varphi$ iff $\mathcal{M}, w \models \neg\Box\neg\varphi$ iff $\mathcal{M}, w \not\models \Box\neg\varphi$ iff it is *not* the case that for all w' with Rww' we have $\mathcal{M}, w' \models \neg\varphi$; i.e. there is some w' with Rww' and $\mathcal{M}, w' \models \varphi$.

Remark 7.2 (Boundary case). If no index is accessible from w (i.e. there is no w' with Rww'), then $\mathcal{M}, w \models \Box\varphi$ holds *vacuously* for every formula φ. Conversely, $\mathcal{M}, w \not\models \Diamond\varphi$ for every φ. This is a direct consequence of the standard treatment of vacuously true universal statements.

7.3 Semantic Concepts

Definition 7.3. Let φ be a formula and Γ a set of formulae.

1. φ is **K-satisfiable** iff there exists a Kripke model $\mathcal{M} = \langle W, R, I\rangle$ and an index $w \in W$ such that $\mathcal{M}, w \models \varphi$.
2. Γ is **K-satisfiable** iff there exist $\mathcal{M}$ and w such that $\mathcal{M}, w \models \gamma$ for every $\gamma \in \Gamma$.
3. An argument from premises $\varphi_1, \ldots, \varphi_n$ to conclusion ψ has a **counter-model** iff the set $\{\varphi_1, \ldots, \varphi_n, \neg\psi\}$ is K-satisfiable.
   It is **K-valid** iff it has no counter-model.
4. φ is a **logical truth of K** (written $\models_K \varphi$) iff $\mathcal{M}, w \models \varphi$ for every Kripke model $\mathcal{M}$ and every index w in $\mathcal{M}$.

Part III
Tableaux for K

8 Rules for $\Box$ (Box)

The tableau rules for $\Box$ follow directly from the truth definition. As in first-order logic, the key distinction is between rules with **universal import** and rules with **existential import**.

8.1 Rule $\mathsf{T}\Box$ (Universal Import)

Rule $\mathsf{T}\Box$

$$\frac{\mathsf{T} : \Box\varphi,\ i \qquad iRj}{\mathsf{T} : \varphi,\ j}$$

where j is an index that **already occurs** on the branch with iRj. This rule may be applied **multiple times** as new accessible indices appear.

If $\Box\varphi$ is true at i, then φ must be true at every index accessible from i. Therefore, whenever we have an accessibility link iRj on the branch, we may add $\mathsf{T} : \varphi, j$. This rule does *not* introduce new indices.

8.2 Rule $\mathsf{F}\Box$ (Existential Import)

Rule $\mathsf{F}\Box$

$$\frac{\mathsf{F} : \Box\varphi,\ i}{\begin{array}{c} iRj \\ \mathsf{F} : \varphi,\ j \end{array}}$$

where j is a **new** index that does **not** yet occur on the branch.

If $\Box\varphi$ is false at i, then there must be some accessible index at which φ fails. We introduce a fresh index j, declare iRj, and record $\mathsf{F} : \varphi, j$.

Remark 8.1. The analogy with first-order logic is exact. The $\mathsf{T}\Box$ rule behaves like $\mathsf{T}\forall$ (universal import: instantiate to all known indices); the $\mathsf{F}\Box$ rule behaves like $\mathsf{F}\forall$ (existential import: introduce a fresh witness).

9 Rules for $\Diamond$ (Diamond)

9.1 Rule $\mathsf{T}\Diamond$ (Existential Import)

Rule $\mathsf{T}\Diamond$

$$\frac{\mathsf{T} : \Diamond\varphi,\ i}{\begin{array}{c} iRj \\ \mathsf{T} : \varphi,\ j \end{array}}$$

where j is a **new** index that does **not** yet occur on the branch.

9.2 Rule $\mathsf{F}\Diamond$ (Universal Import)

Rule $\mathsf{F}\Diamond$

$$\frac{\mathsf{F} : \Diamond\varphi,\ i \qquad iRj}{\mathsf{F} : \varphi,\ j}$$

where j is an index that **already occurs** on the branch with iRj. This rule may be applied **multiple times** as new accessible indices appear.

Remark 9.1. The duality is clean: the $\mathsf{T}\Diamond$ rule is existential (introduces a new index), just as $\mathsf{F}\Box$; the $\mathsf{F}\Diamond$ rule is universal (applies to existing accessible indices), just as $\mathsf{T}\Box$. This reflects the equivalence $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$.

10 Summary of Rules and Strategy

10.1 Summary of Modal Rules

Rule	**Import**	**Index**	**Branching**
$\mathsf{T}\Box$	Universal	Existing j with iRj	No
$\mathsf{F}\Box$	Existential	New j (adds iRj)	No
$\mathsf{T}\Diamond$	Existential	New j (adds iRj)	No
$\mathsf{F}\Diamond$	Universal	Existing j with iRj	No

Remark 10.1. None of the modal rules is branching. Branching only arises from the propositional rules ($\mathsf{F}\wedge$, $\mathsf{T}\vee$, $\mathsf{T}\rightarrow$).

10.2 Order of Rule Application

Procedure for Constructing Modal Tableaux

1. **Apply propositional rules** ($\neg$, $\wedge$, $\vee$, $\rightarrow$) at each index until no more can be applied.
2. **Apply rules with existential import** ($\mathsf{T}\Diamond$ and $\mathsf{F}\Box$); each application introduces a new index. Mark these formulas: they need only be applied once.
3. **Return to step 1**: new propositional material may have appeared at the new indices.
4. **Apply rules with universal import** ($\mathsf{T}\Box$ and $\mathsf{F}\Diamond$) to every index j such that iRj is on the branch. Do **not** mark these formulas: they may need to be re-applied as new accessible indices appear.
5. **Return to step 1**.

A tableau is **finished** when either all branches are closed, or all applicable rules have been exhausted on every open branch.

11 Worked Examples: Proving Validity

11.1 The K-Axiom: $\models_K \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

This is the characteristic axiom of the system K (whence the name). To show it is a logical truth, we attempt to build a counter-model by opening a tableau with its negation.

$$\mathsf{F} : \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q),\ 1\ \checkmark$$

$\mathsf{F}\rightarrow$

$$\mathsf{T} : \Box(p \rightarrow q),\ 1$$
$$\mathsf{F} : \Box p \rightarrow \Box q,\ 1\ \checkmark$$

$\mathsf{F}\rightarrow$

$$\mathsf{T} : \Box p,\ 1$$
$$\mathsf{F} : \Box q,\ 1\ \checkmark$$

$\mathsf{F}\Box$

$$1R2$$
$$\mathsf{F} : q,\ 2$$

$\mathsf{T}\Box$

$$\mathsf{T} : p \rightarrow q,\ 2\ \checkmark$$
$$\mathsf{T} : p,\ 2$$

$\mathsf{T} : q,\ 2$	$\mathsf{F} : p,\ 2$
$\times$	$\times$

Explanation:

1. Apply $\mathsf{F} \to$ to the root: get $\mathsf{T} : \Box(p \to q)$, 1 and $\mathsf{F} : \Box p \to \Box q$, 1.
2. Apply $\mathsf{F} \to$: get $\mathsf{T} : \Box p$, 1 and $\mathsf{F} : \Box q$, 1.
3. Apply $\mathsf{F}\Box$ (existential import) to $\mathsf{F} : \Box q$, 1: introduce index 2, add $1R2$ and $\mathsf{F} : q$, 2.
4. Apply $\mathsf{T}\Box$ (universal import) to $\mathsf{T} : \Box(p \to q)$, 1 and $\mathsf{T} : \Box p$, 1 with $1R2$: get $\mathsf{T} : p \to q$, 2 and $\mathsf{T} : p$, 2.
5. Apply $\mathsf{T} \to$ to $\mathsf{T} : p \to q$, 2: branch into $\mathsf{F} : p$, 2 and $\mathsf{T} : q$, 2.

A contradiction is found on each branch:
- Left: $\mathsf{T} : p$, 2 and $\mathsf{F} : p$, 2. $\times$
- Right: $\mathsf{F} : q$, 2 and $\mathsf{T} : q$, 2. $\times$

All branches are closed, therefore the formula is a **logical truth of** K.

> **The Distribution Axiom.** The schema $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is called the *distribution axiom* (axiom K). Together with the rule of necessitation ("if φ is a theorem, so is $\Box\varphi$"), it characterises the minimal normal modal logic K.

11.2 A Variant: $\models_K \Box(p \to q) \to (\Diamond p \to \Diamond q)$

We verify that necessity distributes analogously over $\Diamond$.

$$\mathsf{F} : \Box(p \to q) \to (\Diamond p \to \Diamond q),\ 1\ \checkmark$$

$\mathsf{F} \to$

$$\mathsf{T} : \Box(p \to q),\ 1$$
$$\mathsf{F} : \Diamond p \to \Diamond q,\ 1\ \checkmark$$

$\mathsf{F} \to$

$$\mathsf{T} : \Diamond p,\ 1\ \checkmark$$
$$\mathsf{F} : \Diamond q,\ 1$$

$\mathsf{T}\Diamond$

$$1R2$$
$$\mathsf{T} : p,\ 2$$

$\mathsf{T}\Box$, $\mathsf{F}\Diamond$

$$\mathsf{T} : p \to q,\ 2\ \checkmark$$
$$\mathsf{F} : q,\ 2$$

$\mathsf{T} : q,\ 2$	$\mathsf{F} : p,\ 2$
$\times$	$\times$

Explanation:

1. Two applications of $\mathsf{F} \to$ yield $\mathsf{T} : \Box(p \to q)$, 1; $\mathsf{T} : \Diamond p$, 1; $\mathsf{F} : \Diamond q$, 1.

2. Apply T$\Diamond$ (existential import): introduce 2, add $1R2$ and T : p, 2.
3. Apply T$\Box$ to T : $\Box(p \to q)$, 1 with $1R2$: get T : $p \to q$, 2.
4. Apply F$\Diamond$ (universal import) to F : $\Diamond q$, 1 with $1R2$: get F : q, 2.
5. Apply T $\to$: branch into F : p, 2 (closes) and T : q, 2 (closes).

All branches close. The formula is a **logical truth of K**. □

11.3 Transitivity: $\Box(p \to q) \wedge \Box(q \to r) \vdash_K \Box(p \to r)$

$$
\begin{array}{l}
\mathsf{T} : \Box(p \to q) \wedge \Box(q \to r),\ 1\ \checkmark \\
\mathsf{F} : \Box(p \to r),\ 1\ \checkmark
\end{array}
$$

T$\wedge$, F$\Box$

$$
\begin{array}{l}
\mathsf{T} : \Box(p \to q),\ 1 \\
\mathsf{T} : \Box(q \to r),\ 1 \\
1R2 \\
\mathsf{F} : p \to r,\ 2\ \checkmark
\end{array}
$$

F $\to$, T$\Box$

$$
\begin{array}{l}
\mathsf{T} : p,\ 2 \\
\mathsf{F} : r,\ 2 \\
\mathsf{T} : p \to q,\ 2\ \checkmark \\
\mathsf{T} : q \to r,\ 2\ \checkmark
\end{array}
$$

$$
\begin{array}{ccc}
 & \mathsf{T} : q,\ 2 & \mathsf{F} : p,\ 2 \\
 & & \times \\
\mathsf{T} : r,\ 2 & & \mathsf{F} : q,\ 2 \\
\times & & \times
\end{array}
$$

Explanation:

1. Apply T$\wedge$: get T : $\Box(p \to q)$, 1 and T : $\Box(q \to r)$, 1.
2. Apply F$\Box$ (existential): introduce 2, $1R2$, F : $p \to r$, 2.
3. Apply F $\to$: T : p, 2 and F : r, 2.
4. Apply T$\Box$ twice with $1R2$: T : $p \to q$, 2 and T : $q \to r$, 2.
5. Apply T $\to$ to $p \to q$: branch into F : p, 2 (closes with T : p, 2) and T : q, 2.
6. Apply T $\to$ to $q \to r$: branch into F : q, 2 (closes with T : q, 2) and T : r, 2 (closes with F : r, 2).

All branches close. The argument is **K-valid**. □

11.4 The Necessitation Rule

Statement. If φ is a logical truth of K (i.e. $\models_K \varphi$), then $\Box\varphi$ is also a logical truth of K (i.e. $\models_K \Box\varphi$).

Semantic argument. Suppose $\models_K \varphi$: for every Kripke model $\mathcal{M}$ and every index w in $\mathcal{M}$, $\mathcal{M}, w \models \varphi$. Now let $\mathcal{M}$ be an arbitrary model and w an arbitrary index; we want to show $\mathcal{M}, w \models \Box\varphi$. By Definition 7.2, this requires that $\mathcal{M}, w' \models \varphi$ for every w' accessible from w. But each such w' is an index in $\mathcal{M}$, and we assumed that φ is true at *every* index in *every* model. So $\mathcal{M}, w' \models \varphi$ holds, and $\mathcal{M}, w \models \Box\varphi$ follows. Since $\mathcal{M}$ and w were arbitrary, $\models_K \Box\varphi$.

Necessitation Rule. If $\models_K \varphi$, then $\models_K \Box\varphi$.

Together with the distribution axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, this rule characterises the minimal normal modal logic K: every theorem of K can be derived from propositional tautologies using modus ponens, the distribution axiom, and the necessitation rule.

11.5 A Common Mistake: Necessitation Is Not a Conditional

Students sometimes confuse the necessitation rule with the schema $\varphi \to \Box\varphi$ ("if φ, then necessarily φ"). The necessitation rule is a *meta-logical* principle: it says that *if φ is a theorem* (true in all models at all indices), then $\Box\varphi$ is also a theorem. It does *not* say that whenever φ happens to be true at some index, $\Box\varphi$ is true at that index.

The schema $p \to \Box p$ is **not** a logical truth of K:

$$\mathsf{F} : p \to \Box p,\ 1\ \checkmark$$

$\mathsf{F}\to$

$$\mathsf{T} : p,\ 1$$
$$\mathsf{F} : \Box p,\ 1\ \checkmark$$

$\mathsf{F}\Box$

$$1R2$$
$$\mathsf{F} : p,\ 2$$

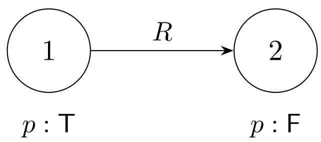
(open)

Explanation:

1. $\mathsf{F}\to$: $\mathsf{T} : p$, 1 and $\mathsf{F} : \Box p$, 1.
2. $\mathsf{F}\Box$ (existential): introduce 2, $1R2$, $\mathsf{F} : p$, 2.
3. No contradiction: $\mathsf{T} : p$, 1 and $\mathsf{F} : p$, 2 concern *different* indices.

The branch is **open**.

Counter-model. $W = \{1, 2\}$, $R = \{\langle 1, 2\rangle\}$, $I(p, 1) = \mathsf{T}$, $I(p, 2) = \mathsf{F}$.



At index 1, p is true, but $\Box p$ is false because p fails at the accessible index 2. So $p \to \Box p$ is false at 1.

Remark 11.1. The distinction is worth memorising: the necessitation rule is an *inference rule* ("from a theorem, derive another theorem"); the schema $\varphi \to \Box\varphi$ is a *formula* ("at any given world, if φ then $\Box\varphi$"). The former is valid in K; the latter is not.

12 Worked Examples: Building Counter-Models

When a finished tableau has at least one *open* branch, the information on that branch can be used to construct a counter-model.

12.1 Recipe for Reading Counter-Models

1. The set of indices W consists of all indices mentioned on the open branch.
2. The accessibility relation R consists of all pairs iRj recorded on the branch.
3. For each propositional variable p and each index i on the branch: set $I(p, i) = \mathsf{T}$ if $\mathsf{T} : p, i$ occurs, and $I(p, i) = \mathsf{F}$ if $\mathsf{F} : p, i$ occurs. (If neither appears, the value may be chosen freely.)

12.2 Example: $\not\models_K (\Box p \to \Box q) \to \Box(p \to q)$

We claim that $(\Box p \to \Box q) \to \Box(p \to q)$ is *not* a logical truth of K.

$\mathsf{F} : (\Box p \to \Box q) \to \Box(p \to q),\ 1\ \checkmark$

$\mathsf{F}\!\to$

$\mathsf{T} : \Box p \to \Box q,\ 1\ \checkmark$
$\mathsf{F} : \Box(p \to q),\ 1\ \checkmark$

$\mathsf{F}\Box$

$1R2$
$\mathsf{F} : p \to q,\ 2\ \checkmark$

$\mathsf{F}\!\to$

$\mathsf{T} : p,\ 2$
$\mathsf{F} : q,\ 2$

$\mathsf{T}\!\to$

$\mathsf{T} : \Box q,\ 1$	$\mathsf{F} : \Box p,\ 1\ \checkmark$
$\mathsf{T}\Box$	$\mathsf{F}\Box$
$\mathsf{T} : q,\ 2$	$1R3$
$\times$	$\mathsf{F} : p,\ 3$
	(open)

Explanation:

1. $\mathsf{F}\!\to$ at root: $\mathsf{T} : \Box p \to \Box q$, 1 and $\mathsf{F} : \Box(p \to q)$, 1.

2. $\mathsf{F}\Box$ (existential): introduce 2, $1R2$, $\mathsf{F} : p \to q$, 2.

3. $\mathsf{F} \to$: $\mathsf{T} : p$, 2 and $\mathsf{F} : q$, 2.

4. $\mathsf{T} \to$ on $\Box p \to \Box q$ at 1: branches into $\mathsf{F} : \Box p$, 1 and $\mathsf{T} : \Box q$, 1.

5. Right sub-branch: $\mathsf{T}\Box$ with $1R2$ gives $\mathsf{T} : q$, 2, contradicting $\mathsf{F} : q$, 2. $\times$

6. Left sub-branch: $\mathsf{F}\Box$ (existential) on $\mathsf{F} : \Box p$, 1: introduce 3, $1R3$, $\mathsf{F} : p$, 3. No contradiction arises. **open**.

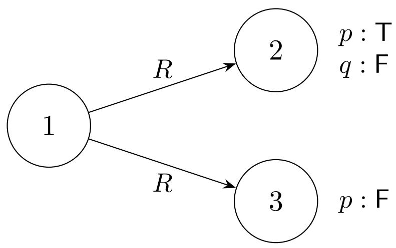
Reading the Counter-Model

Following the open branch, we collect:
- Indices: $1, 2, 3$.
- Accessibility: $1R2$, $1R3$.
- Signed atoms: $\mathsf{T} : p$, 2; $\mathsf{F} : q$, 2; $\mathsf{F} : p$, 3.

The counter-model is:

$$\begin{gathered} W = \{1, 2, 3\}, \\ R = \{\langle 1, 2\rangle,\ \langle 1, 3\rangle\}, \\ I(p, 2) = \mathsf{T}, \quad I(p, 3) = \mathsf{F}, \quad I(q, 2) = \mathsf{F}. \end{gathered}$$

(Values not determined by the branch—$I(q, 1)$, $I(p, 1)$, $I(q, 3)$—can be chosen freely.)



Verification

- Since $I(p, 3) = \mathsf{F}$ and $1R3$, we have $\mathcal{M}, 1 \not\models \Box p$. Therefore $\Box p \to \Box q$ is true at 1 (a conditional with a false antecedent).

- Since $I(p, 2) = \mathsf{T}$ and $I(q, 2) = \mathsf{F}$, we have $\mathcal{M}, 2 \not\models p \to q$. Combined with $1R2$, this gives $\mathcal{M}, 1 \not\models \Box(p \to q)$.

- So at index 1 the antecedent $\Box p \to \Box q$ is true and the consequent $\Box(p \to q)$ is false; the whole conditional is false.

Conclusion: $\not\models_K (\Box p \to \Box q) \to \Box(p \to q)$.

12.3 Necessity Does Not Distribute over Disjunction: $\not\models_K \Box(p \vee q) \to (\Box p \vee \Box q)$

Since $\Box$ distributes over the conditional (the K-axiom), one might expect it to distribute over disjunction as well. It does not: $\Box(p \vee q) \to (\Box p \vee \Box q)$ is *not* a logical truth of K.

$$\mathsf{F} : \Box(p \vee q) \to (\Box p \vee \Box q),\ 1\ \checkmark$$

$\mathsf{F}\to$

$$\mathsf{T} : \Box(p \vee q),\ 1$$
$$\mathsf{F} : \Box p \vee \Box q,\ 1\ \checkmark$$

$\mathsf{F}\vee$

$$\mathsf{F} : \Box p,\ 1\ \checkmark$$
$$\mathsf{F} : \Box q,\ 1\ \checkmark$$

$\mathsf{F}\Box$, $\mathsf{T}\Box$

$$1R2$$
$$\mathsf{F} : p,\ 2$$
$$\mathsf{T} : p \vee q,\ 2\ \checkmark$$

$\mathsf{F}\Box$, $\mathsf{T}\Box$

$$1R3$$
$$\mathsf{F} : q,\ 3$$
$$\mathsf{T} : p \vee q,\ 3\ \checkmark$$

$\mathsf{T}\vee$

$$\mathsf{T} : p,\ 2$$

$\mathsf{T} : q,\ 2$	$\mathsf{F} : p,\ 2$ $\times$

$\mathsf{T}\vee$

$\mathsf{T} : q,\ 3$	$\mathsf{T} : p,\ 3$
$\mathsf{F} : q,\ 3$ $\times$	(open)

Explanation:

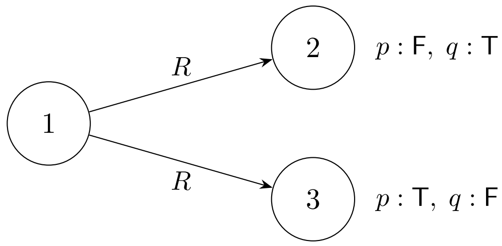
1. $\mathsf{F}\to$: $\mathsf{T} : \Box(p \vee q)$, 1 and $\mathsf{F} : \Box p \vee \Box q$, 1.
2. $\mathsf{F}\vee$ (non-branching): $\mathsf{F} : \Box p$, 1 and $\mathsf{F} : \Box q$, 1.
3. $\mathsf{F}\Box$ on $\mathsf{F} : \Box p$, 1 (existential): introduce 2, $1R2$, $\mathsf{F} : p$, 2.
4. $\mathsf{T}\Box$ on $\mathsf{T} : \Box(p \vee q)$, 1 with $1R2$: $\mathsf{T} : p \vee q$, 2.
5. $\mathsf{F}\Box$ on $\mathsf{F} : \Box q$, 1 (existential): introduce 3, $1R3$, $\mathsf{F} : q$, 3.
6. $\mathsf{T}\Box$ on $\mathsf{T} : \Box(p \vee q)$, 1 with $1R3$: $\mathsf{T} : p \vee q$, 3.
7. $\mathsf{T}\vee$ on $\mathsf{T} : p \vee q$, 2: branch into $\mathsf{T} : p$, 2 (closes with $\mathsf{F} : p$, 2) and $\mathsf{T} : q$, 2.
8. On the surviving branch, $\mathsf{T}\vee$ on $\mathsf{T} : p \vee q$, 3: branch into $\mathsf{T} : p$, 3 (**open**) and $\mathsf{T} : q$, 3 (closes with $\mathsf{F} : q$, 3).

One branch is **open**.

Reading the Counter-Model

The open branch contains: $\mathsf{F} : p,\ 2$; $\mathsf{T} : q,\ 2$; $\mathsf{T} : p,\ 3$; $\mathsf{F} : q,\ 3$.

$$\begin{gathered} W = \{1, 2, 3\}, \\ R = \{\langle 1, 2\rangle,\ \langle 1, 3\rangle\}, \\ I(p, 2) = \mathsf{F}, \quad I(q, 2) = \mathsf{T}, \quad I(p, 3) = \mathsf{T}, \quad I(q, 3) = \mathsf{F}. \end{gathered}$$



Verification

- At index 2, q is true, so $p \vee q$ is true. At index 3, p is true, so $p \vee q$ is true. Since these are all the indices accessible from 1, $\Box(p \vee q)$ is true at 1.
- At index 2, p is false, so $\Box p$ is false at 1.
- At index 3, q is false, so $\Box q$ is false at 1.
- Therefore $\Box p \vee \Box q$ is false at 1, and the whole conditional is false.

Conclusion: $\not\models_K \Box(p \vee q) \rightarrow (\Box p \vee \Box q)$.

Remark 12.1. The counter-model illustrates the key idea: every accessible world can make $p \vee q$ true by choosing a *different* disjunct, so the disjunction is necessarily true without either disjunct being necessarily true.

13 Laws That K Does Not Validate

K is the weakest normal modal logic: it validates only those principles that hold for *every* reading of $\Box$. Principles that are true under some interpretations but not all fail in K.

13.1 The Reflexivity Axiom: $\not\models_K \Box p \rightarrow p$

This schema (known as axiom T) says that whatever is necessary is actual. In the deontic reading, it says that whatever is obligatory is the case—evidently false.

$\mathsf{F} : \Box p \rightarrow p,\ 1$ ✓

$\mathsf{F}\rightarrow$

$\mathsf{T} : \Box p,\ 1$
$\mathsf{F} : p,\ 1$

(open)

No rule forces the pair $1R1$ to appear on the branch (the accessibility relation in K is unconstrained). Without $1R1$, the universal rule $\mathsf{T}\Box$ cannot extract $\mathsf{T} : p,\ 1$ from $\mathsf{T} : \Box p,\ 1$. The branch is **open**.

Counter-model. $W = \{1\}$, $R = \varnothing$, $I(p, 1) = \mathsf{F}$. Since no world is accessible from 1, $\Box p$ is vacuously true at 1; but p is false there.

Remark 13.1. Axiom T corresponds to **reflexivity** of the accessibility relation: $\forall w.\ Rww$. Adding this condition to K yields the modal logic T.

13.2 The Dual: $\not\models_K p \to \Diamond p$

This schema says that whatever is actual is possible.

$$\begin{array}{c} \mathsf{F} : p \to \Diamond p,\ 1 \checkmark \\ \mathsf{F}\to \Big| \\ \mathsf{T} : p,\ 1 \\ \mathsf{F} : \Diamond p,\ 1 \\ \Big| \\ \text{(open)} \end{array}$$

The formula $\mathsf{F} : \Diamond p$, 1 is universal: it requires that for every j with $1Rj$, $\mathsf{F} : p$, j holds. But since no $1Rj$ has been introduced, there is nothing to do. The branch is **open**.

Counter-model. $W = \{1\}$, $R = \varnothing$, $I(p, 1) = \mathsf{T}$. The world 1 satisfies p but sees no world at all, so $\Diamond p$ is false.

13.3 The Schema $\Box p \to \Diamond p$

This schema also fails in K, by the same mechanism.

$$\begin{array}{c} \mathsf{F} : \Box p \to \Diamond p,\ 1 \checkmark \\ \mathsf{F}\to \Big| \\ \mathsf{T} : \Box p,\ 1 \\ \mathsf{F} : \Diamond p,\ 1 \\ \Big| \\ \text{(open)} \end{array}$$

Counter-model. $W = \{1\}$, $R = \varnothing$; the value of $I(p, 1)$ does not matter. $\Box p$ is vacuously true and $\Diamond p$ is false, since no world is accessible.

The counter-model may seem artificial: a world that sees nothing at all. But this situation arises naturally in at least two well-studied interpretations of $\Box$.

Provability logic. Read $\Box \varphi$ as "φ is provable in Peano Arithmetic" and $\Diamond \varphi$ as $\neg\Box\neg\varphi$, i.e. "φ is consistent with PA" (the negation of φ is not provable). Then $\Box p \to \Diamond p$ says: if p is provable, then $\neg p$ is not provable—in other words, PA is consistent. By Gödel's second incompleteness theorem, PA cannot prove its own consistency, so this schema fails in the provability interpretation. The modal logic of provability (GL, Gödel–Löb logic) is an extension of K that validates the Löb axiom $\Box(\Box p \to p) \to \Box p$ but does not validate $\Box p \to \Diamond p$. The Kripke frames for GL are finite, transitive, and irreflexive—they contain "leaf" worlds that see nothing, which is exactly where $\Box p$ holds vacuously while $\Diamond p$ fails.

Temporal logic with a last instant. Read $\Box$ as "it will always be the case that" and $\Diamond$ as "it will at some point be the case that", with the accessibility relation being "lies in the future of". If time has an end—a final moment with no successor—then at that moment $\Box p$ is vacuously true (there is no future instant at which p could fail) and $\Diamond p$ is false (there is no future instant at all). So $\Box p \to \Diamond p$ fails at the last moment.

Remark 13.2. In both cases the failure has the same structure: a world that sees nothing, making the universal claim $\Box p$ vacuously true and the existential claim $\Diamond p$ automatically false. The system D (K + seriality: every world sees at least one world) is the weakest system that rules this out, which is why it is the usual baseline for deontic logic—the assumption that there is always at least one normatively ideal world prevents the absurdity of everything being simultaneously obligatory and nothing being permissible.

Part IV
Conclusion and Outlook

14 Summary

The semantics of K rests on three ingredients: a set of indices (possible worlds), an accessibility relation that determines which worlds are relevant for evaluating modal claims, and an interpretation function that assigns truth values to propositional variables at each world.

1. Intensional contexts motivate the move from extensional to modal logic.
2. The accessibility relation allows the semantics to remain neutral across different interpretations of the modal operator.
3. K is the minimal normal modal logic: it validates the distribution axiom $\Box(\varphi \to \psi) \to (\Box \varphi \to \Box \psi)$ and nothing more (beyond propositional tautologies and closure under necessitation and modus ponens).
4. Tableaux extend straightforwardly: existential rules ($\mathsf{T}\Diamond$, $\mathsf{F}\Box$) introduce new indices; universal rules ($\mathsf{T}\Box$, $\mathsf{F}\Diamond$) propagate information to existing accessible indices.
5. Open branches yield counter-models; closed tableaux establish validity.

In the next lecture, we shall investigate what happens when structural constraints are imposed on the accessibility relation, giving rise to the logics T, $S4$, B, and $S5$.