Solutions to Exercises on Models

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1 Exercise 1

1.1 Show that if $\Gamma \subseteq \Gamma'$, then $\mathrm{Mod}(\Gamma') \subseteq \mathrm{Mod}(\Gamma)$

Suppose for contradiction that $\Gamma \subseteq \Gamma'$ and $\mathrm{Mod}(\Gamma') \nsubseteq \mathrm{Mod}(\Gamma)$. The latter means that there exists a model M such that $M \in \mathrm{Mod}(\Gamma')$ and $M \notin \mathrm{Mod}(\Gamma)$. So M satisfies every formula in Γ', but there exists at least one formula $\phi \in \Gamma$ such that $M \not\models \phi$. This is contradictory: since $\Gamma \subseteq \Gamma'$, we have $\phi \in \Gamma'$, and therefore $M \models \phi$ (because $\phi \in \Gamma'$) and $M \not\models \phi$ (by hypothesis). ■

1.2 Show that $\mathcal{C} \subseteq \mathcal{C}' \Rightarrow \mathrm{Th}(\mathcal{C}') \subseteq \mathrm{Th}(\mathcal{C})$

Assume $\mathcal{C} \subseteq \mathcal{C}'$, and suppose for contradiction that $\mathrm{Th}(\mathcal{C}') \nsubseteq \mathrm{Th}(\mathcal{C})$. This means there exists a formula ϕ such that $\phi \in \mathrm{Th}(\mathcal{C}')$ but $\phi \notin \mathrm{Th}(\mathcal{C})$. Since $\phi \notin \mathrm{Th}(\mathcal{C})$, there exists a model $M \in \mathcal{C}$ such that $M \not\models \phi$. Since $\mathcal{C} \subseteq \mathcal{C}'$, we have $M \in \mathcal{C}'$. This is contradictory: $\phi \in \mathrm{Th}(\mathcal{C}')$ means that ϕ is satisfied by every model in $\mathcal{C}'$, so we have both $M \models \phi$ and $M \not\models \phi$. ■

1.3 Show that $\mathrm{Mod}(\Gamma \cup \Gamma') = \mathrm{Mod}(\Gamma) \cap \mathrm{Mod}(\Gamma')$

We prove this by establishing a double inclusion.

- $\mathrm{Mod}(\Gamma \cup \Gamma') \subseteq \mathrm{Mod}(\Gamma) \cap \mathrm{Mod}(\Gamma')$**.** Suppose for contradiction that there exists a model M such that $M \in \mathrm{Mod}(\Gamma \cup \Gamma')$ and $M \notin \mathrm{Mod}(\Gamma) \cap \mathrm{Mod}(\Gamma')$. If $M \notin \mathrm{Mod}(\Gamma) \cap \mathrm{Mod}(\Gamma')$, then either $M \notin \mathrm{Mod}(\Gamma)$ or $M \notin \mathrm{Mod}(\Gamma')$. Consider the first case: $M \notin \mathrm{Mod}(\Gamma)$. But we assumed $M \in \mathrm{Mod}(\Gamma \cup \Gamma')$, which means M satisfies every formula that belongs to Γ or to Γ'. A fortiori, M satisfies every formula in Γ, so $M \in \mathrm{Mod}(\Gamma)$. Contradiction. The second case ($M \notin \mathrm{Mod}(\Gamma')$) is handled in exactly the same way.
- $\mathrm{Mod}(\Gamma) \cap \mathrm{Mod}(\Gamma') \subseteq \mathrm{Mod}(\Gamma \cup \Gamma')$**.** Suppose $M \in \mathrm{Mod}(\Gamma) \cap \mathrm{Mod}(\Gamma')$. Then $M \in \mathrm{Mod}(\Gamma)$ and $M \in \mathrm{Mod}(\Gamma')$. So M satisfies every formula in Γ and every formula in Γ', hence every formula in $\Gamma \cup \Gamma'$. Therefore $M \in \mathrm{Mod}(\Gamma \cup \Gamma')$.

■

1.4 Show that $\mathrm{Th}(\mathcal{C} \cup \mathcal{C}') = \mathrm{Th}(\mathcal{C}) \cap \mathrm{Th}(\mathcal{C}')$

Again, we establish a double inclusion.

- $\mathrm{Th}(\mathcal{C} \cup \mathcal{C}') \subseteq \mathrm{Th}(\mathcal{C}) \cap \mathrm{Th}(\mathcal{C}')$**.** Suppose for contradiction that there exists a formula ϕ such that $\phi \in \mathrm{Th}(\mathcal{C} \cup \mathcal{C}')$ but $\phi \notin \mathrm{Th}(\mathcal{C}) \cap \mathrm{Th}(\mathcal{C}')$. Then either $\phi \notin \mathrm{Th}(\mathcal{C})$ or $\phi \notin \mathrm{Th}(\mathcal{C}')$. Suppose first that $\phi \notin \mathrm{Th}(\mathcal{C})$. This means there exists a model in $\mathcal{C}$ that falsifies ϕ. But if this model belongs to $\mathcal{C}$, then a fortiori it belongs to $\mathcal{C} \cup \mathcal{C}'$. Since $\phi \in \mathrm{Th}(\mathcal{C} \cup \mathcal{C}')$, ϕ is satisfied by every element of $\mathcal{C} \cup \mathcal{C}'$. Contradiction. If instead $\phi \notin \mathrm{Th}(\mathcal{C}')$, the reasoning is identical.

- $\mathrm{Th}(\mathcal{C}) \cap \mathrm{Th}(\mathcal{C}') \subseteq \mathrm{Th}(\mathcal{C} \cup \mathcal{C}')$**.** Suppose for contradiction that there exists a formula ϕ such that $\phi \in \mathrm{Th}(\mathcal{C}) \cap \mathrm{Th}(\mathcal{C}')$ and $\phi \notin \mathrm{Th}(\mathcal{C} \cup \mathcal{C}')$. From $\phi \in \mathrm{Th}(\mathcal{C})$ and $\phi \in \mathrm{Th}(\mathcal{C}')$, ϕ is true in every model belonging to $\mathcal{C}$ and in every model belonging to $\mathcal{C}'$. Therefore ϕ is true in every model that belongs to $\mathcal{C}$ or to $\mathcal{C}'$. So $\phi \in \mathrm{Th}(\mathcal{C} \cup \mathcal{C}')$. Contradiction.

■

1.5 Show that $\mathcal{C} \subseteq \mathrm{Mod}(\Gamma) \Leftrightarrow \Gamma \subseteq \mathrm{Th}(\mathcal{C})$

We prove both directions of the biconditional separately.

- ($\Rightarrow$) Suppose $\mathcal{C} \subseteq \mathrm{Mod}(\Gamma)$ but $\Gamma \nsubseteq \mathrm{Th}(\mathcal{C})$. Then there exists a formula $\phi \in \Gamma$ and a model $M \in \mathcal{C}$ such that $M \not\models \phi$. But by hypothesis, $M \in \mathrm{Mod}(\Gamma)$. Contradiction, since M would then have to both satisfy and falsify ϕ.
- ($\Leftarrow$) Suppose $\Gamma \subseteq \mathrm{Th}(\mathcal{C})$ but $\mathcal{C} \nsubseteq \mathrm{Mod}(\Gamma)$. Then there exists a model $M \in \mathcal{C}$ such that $M \notin \mathrm{Mod}(\Gamma)$. So there exists a formula $\phi \in \Gamma$ such that $M \not\models \phi$. But since $\phi \in \Gamma$, we also have $\phi \in \mathrm{Th}(\mathcal{C})$. Since $M \in \mathcal{C}$, it follows that $M \models \phi$. Contradiction.

■

2 Exercise 2

2.1 Show that $\Gamma \subseteq \mathrm{Th}(\mathrm{Mod}(\Gamma))$

Suppose for contradiction that this is not the case. Then there exists a formula $\phi \in \Gamma$ such that $\phi \notin \mathrm{Th}(\mathrm{Mod}(\Gamma))$. This implies that there is a model $M \in \mathrm{Mod}(\Gamma)$ such that $M \not\models \phi$. But by definition, $\mathrm{Mod}(\Gamma) = \{\mathcal{M} \mid \forall \psi \in \Gamma,\ \mathcal{M} \models \psi\}$. Since $\phi \in \Gamma$, we have both $M \models \phi$ and $M \not\models \phi$. Contradiction. ■

2.2 Show that $\mathcal{C} \subseteq \mathrm{Mod}(\mathrm{Th}(\mathcal{C}))$

Suppose for contradiction that this is not the case. Then there exists a model $M \in \mathcal{C}$ such that $M \notin \mathrm{Mod}(\mathrm{Th}(\mathcal{C}))$. This implies that there exists a formula $\phi \in \mathrm{Th}(\mathcal{C})$ such that $M \not\models \phi$. But by definition, $\mathrm{Th}(\mathcal{C}) = \{\phi \in \mathcal{L} \mid \forall \mathcal{M} \in \mathcal{C},\ \mathcal{M} \models \phi\}$. Since $M \in \mathcal{C}$, our initial hypothesis leads to a contradiction. ■

2.3 Show that $\mathrm{Th}(\mathrm{Mod}(\Gamma))$ is a theory axiomatised by Γ

Suppose for contradiction that this is not the case. Then there exists a formula $\phi \in \mathrm{Th}(\mathrm{Mod}(\Gamma))$ such that $\Gamma \nvdash \phi$, and therefore (by soundness) $\Gamma \not\models \phi$. This means there exists a model M that satisfies every formula in Γ but such that $M \not\models \phi$. Since M satisfies every formula in Γ, we have $M \in \mathrm{Mod}(\Gamma)$. But $\phi \in \mathrm{Th}(\mathrm{Mod}(\Gamma))$, so ϕ is satisfied by every model in $\mathrm{Mod}(\Gamma)$, in particular by M. Therefore ϕ is both satisfied and falsified by M. Contradiction. ■

2.4 A theory is said to be *maximally consistent* if (i) the theory is consistent (i.e. no contradiction can be derived from it) and (ii) adding any formula to the theory makes it inconsistent: if Γ is maximally consistent and $\phi \notin \Gamma$, then $\Gamma \cup \{\phi\} \vdash \bot$. Show that Γ is maximally consistent if and only if Γ is complete and consistent.

We prove a biconditional, so we proceed in two steps.

2.4.1 If Γ is maximally consistent, then Γ is complete and consistent.

Assume Γ is maximally consistent. By definition it is consistent. It remains to show that it is complete. We reason by cases, for an arbitrary sentence ϕ in the language of the theory:

- Suppose $\phi \notin \Gamma$. Since the theory is maximal, $\Gamma \cup \{\phi\} \vdash \bot$. By the rule of negation introduction (from a finite subset of Γ), we obtain $\Gamma \vdash \neg\phi$. Since Γ is a theory (deductively closed), $\neg\phi \in \Gamma$.
- Suppose $\neg\phi \notin \Gamma$. By the same reasoning, we can derive $\neg\neg\phi$ from Γ, and therefore $\phi \in \Gamma$.

So if $\phi \notin \Gamma$, then $\neg\phi \in \Gamma$; and if $\neg\phi \notin \Gamma$, then $\phi \in \Gamma$. In either case, $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. This means Γ is complete. ■

2.4.2 If Γ is complete and consistent, then Γ is maximally consistent.

It suffices to show that completeness implies maximality. Consider an arbitrary formula ϕ in the language of the theory. Since Γ is complete, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. So if $\phi \notin \Gamma$, then $\neg\phi \in \Gamma$. This means $\Gamma \cup \{\phi\}$ contains both ϕ and $\neg\phi$, and therefore $\Gamma \cup \{\phi\} \vdash \bot$. Completeness thus implies maximality. ■

3 Exercise 3

3.1 Show that $\mathrm{Mod}(\{\phi_1, \dots, \phi_n\}) = \mathrm{Mod}(\{\phi_1 \wedge \cdots \wedge \phi_n\})$

We show that the two sets have exactly the same elements. We have $M \in \mathrm{Mod}(\{\phi_1, \dots, \phi_n\})$ if and only if $M \models \phi_1$ and ... and $M \models \phi_n$, if and only if $M \models \phi_1 \wedge \cdots \wedge \phi_n$. Therefore $M \in \mathrm{Mod}(\{\phi_1, \dots, \phi_n\})$ if and only if $M \in \mathrm{Mod}(\{\phi_1 \wedge \cdots \wedge \phi_n\})$, and so $\mathrm{Mod}(\{\phi_1, \dots, \phi_n\}) = \mathrm{Mod}(\{\phi_1 \wedge \cdots \wedge \phi_n\})$. ■

3.2 Let Γ be an infinite set of formulas and $\Delta \subset \Gamma$ a finite subset. Using the completeness theorem, show that: $\Gamma \models \phi$ implies that there exists a finite $\Delta \subset \Gamma$ such that $\Delta \models \phi$.

This is fairly immediate. By the completeness theorem, $\Gamma \models \phi$ implies $\Gamma \vdash \phi$. This means there is a proof of ϕ from the set of premises Γ. But any proof uses only finitely many premises. So there exists a finite subset $\Delta \subseteq \Gamma$ of premises that are actually used in the proof. Therefore $\Delta \vdash \phi$. By soundness, $\Delta \models \phi$. ■

3.3 Show that a consistent theory Γ is complete if and only if (i) it has at least one model and (ii) all its models are elementarily equivalent.

A preliminary reminder: two $\mathcal{L}$-structures $\langle M, I\rangle$ and $\langle N, J\rangle$ are *elementarily equivalent* if and only if they satisfy exactly the same first-order sentences. We proceed in two steps.

3.3.1 If a consistent set of sentences Γ is complete, then it has at least one model and all its models are elementarily equivalent.

First, the difficult lemma implies that if Γ is consistent, then it is satisfiable, and therefore has at least one model. (We assumed the difficult lemma in the lectures, so this part is immediate.)

Suppose for contradiction that Γ has two models M and N that are not elementarily equivalent. Then there exists a sentence ϕ such that $M \models \phi$ and $N \not\models \phi$, i.e. $N \models \neg\phi$. Since Γ is complete, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. If $\phi \in \Gamma$, then since N satisfies Γ we have $N \models \phi$. Contradiction. If $\neg\phi \in \Gamma$, then since M satisfies Γ we have $M \models \neg\phi$. Contradiction again. By reductio, all models of a complete consistent theory are elementarily equivalent. ■

3.3.2 If a theory Γ has at least one model and all its models are elementarily equivalent, then it is consistent and complete.

First, if Γ is satisfiable then Γ is consistent. Indeed, $\perp$ is true in no model by definition; if Γ were inconsistent then $\Gamma \vdash \perp$, and by soundness $\Gamma \models \perp$, meaning every model of Γ is a model of $\perp$—but $\perp$ has no models at all.

Now suppose Γ has at least one model and all its models are elementarily equivalent. Consider an arbitrary sentence ϕ in the language of the theory, and an arbitrary model M satisfying Γ. Either $M \models \phi$ or $M \models \neg\phi$. In the first case, ϕ is true in every model of Γ (by elementary equivalence), so $\Gamma \models \phi$, and therefore $\phi \in \Gamma$. In the second case, $\neg\phi$ is true in every model of Γ (same reason), so $\Gamma \models \neg\phi$, and therefore $\neg\phi \in \Gamma$. In either case, $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. This shows Γ is complete. ■

4 Exercise 4: Show that the completeness theorem implies the difficult lemma

By the completeness theorem, $\Gamma \models \phi \Rightarrow \Gamma \vdash \phi$. Now suppose Γ is not satisfiable; we aim to show that Γ is inconsistent. If Γ is not satisfiable, then no model satisfies every formula in Γ, so $\Gamma \models \perp$. By the completeness theorem, $\Gamma \vdash \perp$. This establishes that Γ is inconsistent.

Taking the contrapositive: if Γ is consistent, then Γ is satisfiable. This is precisely the difficult lemma. ■