Exercises on Models

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Exercise 1

1. Show that if $\Gamma \subseteq \Gamma'$, then $\mathrm{Mod}(\Gamma') \subseteq \mathrm{Mod}(\Gamma)$.

2. Show that $\mathcal{C} \subseteq \mathcal{C}' \;\Rightarrow\; \mathrm{Th}(\mathcal{C}') \subseteq \mathrm{Th}(\mathcal{C})$.

3. Show that $\mathrm{Mod}(\Gamma \cup \Gamma') = \mathrm{Mod}(\Gamma) \cap \mathrm{Mod}(\Gamma')$.

4. Show that $\mathrm{Th}(\mathcal{C} \cup \mathcal{C}') = \mathrm{Th}(\mathcal{C}) \cap \mathrm{Th}(\mathcal{C}')$.

5. Show that $\mathcal{C} \subseteq \mathrm{Mod}(\Gamma) \;\Leftrightarrow\; \Gamma \subseteq \mathrm{Th}(\mathcal{C})$.

Exercise 2

1. Show that $\Gamma \subseteq \mathrm{Th}(\mathrm{Mod}(\Gamma))$.

2. Show that $\mathcal{C} \subseteq \mathrm{Mod}(\mathrm{Th}(\mathcal{C}))$.

3. Show that $\mathrm{Th}(\mathrm{Mod}(\Gamma))$ is a theory axiomatised by Γ.

4. A theory is said to be *maximally consistent* if (i) it is consistent (i.e. no contradiction can be derived from it) and (ii) adding any formula to it makes it inconsistent: if Γ is maximally consistent and $\phi \notin \Gamma$, then $\Gamma \cup \{\phi\} \vdash \perp$.

Show that Γ is maximally consistent if and only if Γ is complete and consistent.

Exercise 3

1. Show that $\mathrm{Mod}(\{\phi_1, \ldots, \phi_n\}) = \mathrm{Mod}(\{\phi_1 \wedge \cdots \wedge \phi_n\})$.

2. Let Γ be an infinite set of formulas. Using the completeness theorem, show that if $\Gamma \models \phi$, then there exists a finite $\Delta \subseteq \Gamma$ such that $\Delta \models \phi$.

3. Show that a consistent theory Γ is complete if and only if (i) it has at least one model and (ii) all its models are elementarily equivalent.

Recall: Two $\mathcal{L}$-structures $\langle M, I\rangle$ and $\langle N, J\rangle$ are *elementarily equivalent* if and only if they satisfy exactly the same first-order sentences.

Exercise 4

Show that the completeness theorem implies the difficult lemma (i.e. that if Γ is consistent, then Γ is satisfiable).