Theories and models

I Axiomatized theories

Theories

- A set of sentences Γ is **semantically closed** iff: if $\Gamma \vDash \phi$, then $\phi \in \Gamma$.

    - It is **syntactically closed** iff if $\Gamma \vdash \phi$, then $\phi \in \Gamma$. Because FOL is complete, Γ is semantically closed iff it is syntactically closed. A deductively closed set of formulas is called a **theory**.

- The **semantic closure** of a set of sentences Γ is $\{\phi \mid \Gamma \vDash \phi\}$; the syntactic closure is $\{\phi \mid \Gamma \vdash \phi\}$. Because FOL is complete, $\{\phi \mid \Gamma \vDash \phi\}$=$\{\phi \mid \Gamma \vdash \phi\}$.

- When a set of formulas Γ is the closure of another set Δ (so with $\Delta \subseteq \Gamma$), we'll say that Γ is a **theory axiomatized** by Δ.

- First-order logic is a great tool to express a domain of knowledge in an axiomatized way.

- Given a first-order language with non-logical symbols (=signature) for the primitives of the axiomatically developed science we wish to study, together with a set of sentences that express the fundamental laws of the science, **we can first think of the theory as represented by all the sentences in this language that are entailed by the axioms**, that is, the **deductive closure** of the axioms.

- But there is another, more « model-theoretic » way to think about a theory: **as the set of sentences satisfied by a given $\mathscr{L}$-structure (or model).**

Non-trivial axiomatization

- But we should be careful: trivially, a closed theory is its own axiomatization …
- … but if the theory is infinite, this might not be very useful.
- So we'll say that a theory is **non-trivially axiomatizable** iff:
  - It is axiomatized by Δ
  - Δ is decidable iff there is an **effective procedure** to determine whether a sentence of the langage is a member of the set or not.
  - We'll consider that a method is effective iff it is algorithmic, that is, if it can by mechanically carried out in a finite number of steps.

Theory of a model

- Let us consider a $\mathscr{L}$-structure $\mathscr{M}$. Let us define $Th(\mathscr{M})$ as : $\{\phi \in \mathscr{L} \mid \mathscr{M} \vDash \phi\}$. We'll call this set the **theory of model** $\mathscr{M}$, or the **set of theorems of** $\mathscr{M}$.

- **This set is indeed a theory, because it is closed**— as we can prove. Let $Th(\mathscr{M})$⊨ ψ; then all the models that satisfies all the formulas in $Th(\mathscr{M})$ satisfy ψ; by definition, this is the case for $\mathscr{M}$; so $\mathscr{M} \vDash \psi$, hence $\psi \in \ Th(\mathscr{M})$ and $Th(\mathscr{M})$ is closed.

- Let us now consider a class of structures $\mathscr{C}$. We can define the theory of $\mathscr{C}$, $Th(\mathscr{C})$= $\{\phi \in \mathscr{L} \mid \forall \mathscr{M} \in \mathscr{C}, \mathscr{M} \vDash \phi\}$.

    - Exercise: prove that $Th(\mathscr{C})$ is closed.

Complete theories

- A theory Γ is **complete** iff given any sentence ϕ in the language of the theory, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.

- Let us consider any model $\mathscr{M}$. **The theory of $\mathscr{M}$, $Th(\mathscr{M})$, is complete.**

  - If not, there is at least a sentence ϕ in the language of the theory, such that $\phi \notin Th(\mathscr{M})$ and $\neg\phi \notin Th(\mathscr{M})$. But that's impossible: if $\phi \notin Th(\mathscr{M})$, $\mathscr{M} \nvDash \phi$ and so $\mathscr{M} \vDash \neg\phi$; and if $\neg\phi \notin Th(\mathscr{M})$, $\mathscr{M} \nvDash \neg\phi$, and so $\mathscr{M} \vDash \phi$

- **CAUTION**: you should not confuse the concept of a complete formal system (or logic) with the concept of a complete theory!

A first (trivially) complete theory

- Let us consider the inconsistent theory, axiomatized by $\exists x(x \neq x)$.

- The axiomatization is non-trivial since there is only one axiome.

- It’s a very special theory, that doesn’t have any model. It contains all the sentences of the language.

- Hence, it is complete, but trivially so.

How to prove the incompleteness of an axiomatization?

- One easy method is to find a formula ϕ t**hat cannot be proved nor refuted** from the axioms Δ.
  - Cannot be proved: **find a model of the axioms that falsify the formula**
  - Cannot be refuted: **find a model of the axioms that falsify the negation of the formula**

How to prove the independence of an axiom?

- Given a set of axioms Δ and a new axiom ϕ, we want to show that ϕ can **neither be proved nor refuted** from Δ: $\Delta \nvdash \phi$ and $\Delta \nvdash \neg\phi$.

- Contrapositive of soundness: $\Delta \nvDash \phi \Rightarrow \Delta \nvdash \phi$

- Model-theoretic method: find a counter-model, that is, a structure that satisfies the old axioms but not the new one; and another (maybe distinct) structure that satisfies the old axioms but not the negation of the new one.

How to prove consistency?

- Reminder:
  - Γ is consistent iff $\Gamma \nvdash \bot$
  - Contrapositive of soundness: if $\Gamma \nvDash \bot$ then $\Gamma \nvdash \bot$
- So all we have to show it to prove that there exists a structure (i) that satisfies Γ and (ii) that does not satisfy $\bot$. But nothing satisfies $\bot$ anyway, so we just have to find a structure satisfying Γ.
- To sum up: if Γ **satisfiable, it is consistent.**
- **CAUTION: we have not proved the converse yet, that is, that if Γ is consistent, it is satisfiable! It's a difficult theorem to prove actually because it's equivalent to completeness …**

II Examples of first-order theories

Group theory

- Signature: $\{1, \bullet\}$, where 1 is a constant symbol and $\bullet$ a two-place function symbole.

- Axioms:

  - $\forall x \exists y (x \bullet y = 1)$: every element has an inverse.

  - $\forall x (x \bullet 1 = x)$: identity element

  - $\forall x \forall y \forall z (x \bullet (y \bullet z) = (x \bullet y) \bullet z)$: associativity

Group theory in not complete

- Let us consider the following sentence in the language of group theory:

    - $\forall x \forall y((x \bullet y) \rightarrow (y \bullet x))$: commutativity

- It is easy to find $\mathscr{L}$-structures that satisfies all the axioms of group theory and commutativity, as well as structures that satisfies all the axioms of group theory but falsifies commutativity.

    - Cf Wikipedia, Abelian and non-abelian groups.

- NB : in algebra, many theories are incomplete, because the aim of algebraic theories is to characterize class of structures in an abstract way, not to pin-down a single individual structure.

Robinson's arithmetics Q

1. $\forall x(0 \neq x')$
2. $\forall x \forall y((x' = y') \rightarrow (x = y))$
3. $\forall x(x \neq 0 \rightarrow \exists y(x = y'))$
4. $\forall x(x + 0 = x)$
5. $\forall x \forall y(x + y') = (x + y)'$
6. $\forall x(x.0 = 0)$
7. $\forall x \forall y(x\,.\,y' = (x\,.\,y) + x)$

Robinson's arithmetics is not complete

- For instance, $\forall x(0 + x = x)$ is independent from Q's axioms, it can be neither proved nor refuted from the axioms.
    - Idea of the proof: find a non-standard model of Q that falsifies $\forall x(0 + x = x)$; in such a structure « + » should not be interpreted as the standard addition, in order for there to be counter-examples to $\forall x(0 + x = x)$.

Peano’s arithmetics

1. $\forall x(0 \neq x')$
2. $\forall x \forall y((x' = y') \to (x = y))$
3. $\forall x(x + 0 = x)$
4. $\forall x \forall y(x + y') = (x + y)'$
5. $\forall x(x.0 = 0)$
6. $\forall x \forall y(x \,.\, y' = (x \,.\, y) + x)$
7. All axioms of form: $(\phi(0) \wedge \forall x(\phi(x) \to \phi(x'))) \to \forall x \phi(x)$, where ϕx is a formula where x occurs free, and $\phi(0)$ an instance where these free occurrences are replaced by 0.

Peano's arithmetic PA is not complete

- This fact is much less intuitive than the incompleteness of Q, and more difficult to prove.

- It was proven by Kurt Gödel in 1931.

- If we are confident that there exists a standard, intended, model for arithmetics, $(\mathbb{N}, 0, ', +, .)$, we see that $Th(\mathbb{N}, 0, ', +, .) \neq$ PA

- Philosophical discussion: Gödel result shows that we can come to know some sentences that are true in $(\mathbb{N}, 0, ', +, .)$, but that do not belong to PA, and so that cannot be derived from its axioms.

  - Should be conclude that in a way, mathematical thought is not mechanizable?

Remarks

- An axiomatized theory Δ exactly characterizes a class of structures $\mathscr{C}$ when it is identical to the theorems of this class, that is iff $\Delta = Th(\mathscr{C})$. (???)

- An axiomatized theory Δ exactly characterizes a unique structure when it is identical to the theorems of this structure.

III Characterizing the size of structures

At least n

- There are some properties of structures that can be expressed by using only the purely logical vocabulary of FOL.

- Let us consider for instance the following sentence:

    - $\exists x \exists y (x \neq y)$

- This sentence is true in $\mathscr{M}$ iff $\mathscr{M}$'s domain has at least two elements: **it forces the domain to have at least 2 elements.**

- This generalizes to n: let us indeed consider :

    - ϕ_n: $\exists x_1 \exists x_2 \ldots \exists x_n ((x_1 \neq x_2) \wedge (x_1 \neq x_3) \ldots \wedge (x_{n-1} \neq x_n))$

- For a given n, ϕ_n forces the domain to have at least n éléments.

At most n, exactly n

- Let us consider: $\forall x \forall y \forall z (x = y \vee x = z \vee y = z)$
- It forces a domain to have at most 2 elements.
- $\psi_n : \forall x_1 \forall x_2 \ldots \forall x_{n+1} (x_1 = x_2 \vee x_1 = x_3 \ldots \vee x_n = x_{n+1})$
- ψ_n forces the domain of a structure to have at most n elements
- So $\theta_n = \ \phi_n \wedge \psi_n$ forces the domain of a structure to have exactly n elements

Infinite domains

- Let us now consider the set Δ containing $\{\theta_n \mid n \in \mathbb{N}\}$
- There is an infinite number of formulas in the set, but it is decidable since there is an effective procedure to check whether or not a FOL formula belongs to the set.
- $M \models \Delta$ iff M's domain is infinite.
- On the other hand, as we will see later, the property « having a finite domain » cannot be expressed in first-order logic.

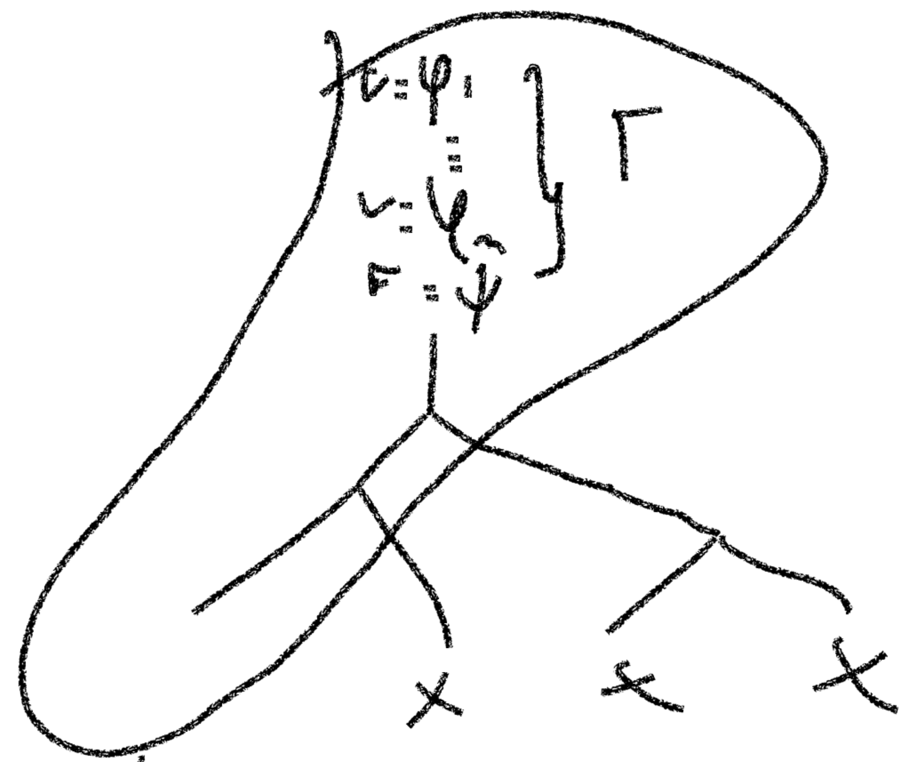
IV Completeness

Completeness for tableaux: A holistic proof

- The proof of soundness in atomistic: we did prove that each rule « conserve truth »; in the case of tableaux: if a branch is satisfiable, expanding it by applying a rule leaves the tree open.
- The proof of completeness has to be more holistic: we want to show that the derivation system taken as a whole is complete.
- Contrapositive: $\Gamma \vDash \theta \Rightarrow \Gamma \vdash \theta$ iff $\Gamma \nvdash \phi \Rightarrow \Gamma \nvDash \theta$
- So: if a tree remains open, then $\Gamma, \neg\phi$ is satisfiable.
- This means that we'd like to prove that when a finished tree for $\Gamma, \neg\phi$ remains open, one can build a model for $\Gamma, \neg\phi$.
- We should prove that any **finished open branch of a tableau is satisfiable.**

- If all the sentences along a branch Δ are satisfiable, then a fortiori (by semantic monotony since $\Gamma \subseteq \Delta$), Γ is satisfiable

- NB : while building the tree we always expand Γ into a bigger set of sentences including it.



At least one branch has to be satisfiable!

Strategy

- Saying that the sentences along a branch are satisfiable amounts to saying that there is a model $\mathscr{M}$ such that: $\mathscr{M}$ satisfies all the true sentences, and falsifies all the false sentences.

- Two tasks:

  - From the information on the branch, build a model $\mathscr{M}$

  - Show that $\mathscr{M}$ satisfies all the true sentences, and falsifies all the false sentences.

- Fortunately, we already know that we can build a model from an open branch!

Building a term model: The idea

- To make the proof more accessible we'll restrict ourselves to FOL without identity, without function symbols, and without quantifiers.

- How can we build a model from a mere set of formulas?

  - Thought: let's consider the proper names occurring along a branch as the elements of the domain; and let's build the model so that the true atomic formulas are true in it, and the false atomic formulas false in it, as we do in the exercises.

Building a term model: Let's do it!

- **Definition**: let us suppose an open branch B in a finished tableau. Let $\mathscr{C} = \{a_1 \dots a_n\}$ the set of constants occurring in the atomic formulas found along B. We call « term model induced by B » the model $\mathscr{M} = < \mathscr{C}, \mathscr{I} >$ such that:
  - The domain of discourse is the set of all constants on branch B, $\mathscr{C}$
  - The interpretation function $\mathscr{I}$ is defined thus. For any atomic sentence $Ra_1 \dots a_n$ occurring on B:
    - If $T : Ra_1 \dots a_n$ then $< a_1 \dots a_n > \in \mathscr{I}(R)$
    - If $F : Ra_1 \dots a_n$ then $< a_1 \dots a_n > \notin \mathscr{I}(R)$

Inductive proof: first step

- Let us sketch an **inductive proof** to the effect that : if B is a finished open branch of a tableau, then B is satisfiable.

- The induction is on the **complexity of formulas on B**.

- **First step of the induction**: if $Ra_1 \ldots a_n$ is any atomic formula, let us consider the term model $\mathscr{M}$ induced by B. By definition: if $T : Ra_1 \ldots a_n$ then $< a_1 \ldots a_n > \in \mathscr{I}(R)$, therefore $\mathscr{M} \vDash Ra_1 \ldots a_n$; and if $F : Ra_1 \ldots a_n$ then $< a_1 \ldots a_n > \notin \mathscr{I}(R)$, therefore $\mathscr{M} \nvDash Ra_1 \ldots a_n$

- **Inductive hypothesis**: if a formula ϕ has complexity n, and if it is on B, then : if $T : \phi$ then $\mathscr{M} \vDash \phi$; and if $F : \phi$ then $\mathscr{M} \nvDash \phi$

Inductive argument

- Let us consider a formula θ of complexity n+1.
- θ has one of the following forms:
  - $\neg\phi$
  - $\phi \wedge \psi$
  - $\phi \vee \psi$
  - $\phi \rightarrow \psi$

- If $\theta = \neg\phi$:

    - If $T : \neg\phi$ is on the branch, since the branch is finished, $F : \phi$ is also on the branch. ϕ has complexity n. So by the inductive hypothesis: $\mathscr{M} \nvDash \phi$, hence $\mathscr{M} \vDash \neg\phi$, hence $\mathscr{M} \vDash \theta$

    - If $F : \neg\phi$ is on the branch, since the branch is finished, $T : \phi$ is also on it. By the inductive hypothesis: $\mathscr{M} \vDash \phi$, hence $\mathscr{M} \nvDash \theta$.

- If $\theta = \phi \wedge \psi$:
  - If $T : \phi \wedge \psi$ is on the branch, since the branch is finished $T : \phi$ and $T : \psi$ are also on the branch. So $\mathscr{M} \vDash \phi$ and $\mathscr{M} \vDash \psi$. Hence, $\mathscr{M} \vDash \phi \wedge \psi$; which is what we wanted to show.
  - $F : \phi \wedge \psi$, a branching rule has been applied: either $F : \phi$ (left branch) or $F : \psi$ (right branch). In the first case, $\mathscr{M} \nvDash \phi$, and (a fortiori) $\mathscr{M} \nvDash \phi \wedge \psi$. In the second case, $\mathscr{M} \nvDash \psi$ and (a fortiori) $\mathscr{M} \nvDash \phi \wedge \psi$. So whatever the case, $\mathscr{M} \nvDash \phi \wedge \psi$, which is what we wanted to show.

Problem

- Let consider the case where $\theta = \forall x \phi x$, and where the formula $T : \forall x \phi x$ appears on a branch, and try to show that $\mathscr{M} \vDash \forall x \phi x$.
    - Hint: remember that the universally quantified formula has to be instantiated to all the constants occurring along the branch in order for the tree to be finished.

The completeness theorem and the difficult lemma

- Reminder: A derivation system is complete iff given any set of sentences Γ:
  - $\Gamma \vDash \theta \Rightarrow \Gamma \vdash \theta$
- Difficult lemma: any set of sentences Γ is such that:
  - If Γ is consistent, then it is satisfiable.
- The completeness theorem and the difficult lemma are logically equivalent.

Proof

- Completeness follows from the difficult lemma:
  - If Γ is consistent, then it is satisfiable. By contraposition: if Γ is not satisfiable, it is inconsistent. Let us now suppose that $\Gamma \vDash \theta$. $\Gamma \cup \{\neg\theta\} \vDash \bot$, which means that $\Gamma \cup \{\neg\theta\}$is not satisfiable, hence not consistent. So $\Gamma \cup \{\neg\theta\} \vdash \bot$ and by introducing the negation and eliminating the double negation we get : $\Gamma \vdash \theta$.
- The difficult lemma follows from completeness:
  - $\Gamma \vDash \theta \Rightarrow \Gamma \vdash \theta$. Let us just replace θ with $\bot$ and we get: $\Gamma \vDash \bot \Rightarrow \Gamma \vdash \bot$. Let us now take the contrapositive: $\Gamma \nvdash \bot \Rightarrow \Gamma \nvDash \bot$. Which means that if Γ is consistent, then it is satisfiable.