Inductive proofs

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1 The main idea

We can reason inductively from the moment we can enumerate items (numbers, formulas, symbols, etc.), and more generally when we can consider the successor of an item. In the case of logical symbols, it is possible to order them in relation to their complexity: atomic formulas are considered to be of zero complexity, then a complex formula is considered to be of degree n + 1 complexity in relation to a formula of degree n if it is constructed by a grammatical rule from this formula. For instance in the language of propositional logic, p has a 0 complexity, $\neg p$ a complexity of 1, $\neg p \vee q$ of 2, etc.
There are always three steps in an inductive proof:

- base case: we check that the fact we want to prove is true at the first rank in our enumeration;
- inductive assumption: we assume the fact we want to prove to be true at an arbitrary rank n;
- inductive argument: given the inductive assumption, we show it is true at rank $n + 1$

2 illustration

As an illustration, let's show that first-order well-formed formulas have as many left-parenthesis as right-parenthesis.

1. base case: no parenthesis occurs in an identity statement $s = t$; and in a relational atomic statement $R(t_1...t_n)$ there is one left-parenthesis and one right-parenthesis
2. inductive assumption: we assume that a formula of complexity n has as many left-parenthesis as right-parenthesis;
3. inductive argument: a complex formula θ can be built from less complex ones only in the following way:

- Negation: $\theta = \neg(\psi)$. We add one left-parenthesis and one right-parenthesis. Since we know, because of the inductive hypothesis, that our theorem is true for ψ, it is also true for θ
- Conjunction, disjunction, conditional: $\theta = (\phi \star \psi)$, where $\star$ is one of the propositional connectors. Because of the inductive assumption, as many left-parenthesis as right-parenthesis occur in ϕ and ψ; to build θ, we add to that one left-parenthesis and one right-parenthesis. So the theorem is true for θ.
- Quantifiers: $\theta = \forall x(\phi x)$ or $\theta = \exists x(\phi x)$. Again, we add one left-parenthesis and one right-parenthesis, so by the inductive assumption the theorem is true for θ.