Solutions: Natural Deduction for First-Order Logic

1 Quantification

1.1 $\forall x \neg Fx \vdash \exists x(Fx \to Gx)$

Strategy. From the premise we obtain $\neg Fa$; assuming Fa produces a contradiction, from which Ga follows by $\bot$E. Discharging yields $Fa \to Ga$, whence $\exists x(Fx \to Gx)$ by $\exists$I.

1	$\forall x \neg Fx$	
2	$\neg Fa$	$\forall$E, 1
3	$\mid\ Fa$	
4	$\mid\ \bot$	$\neg$E, 2, 3
5	$\mid\ Ga$	$\bot$E, 4
6	$Fa \to Ga$	$\Rightarrow$I, 3–5
7	$\exists x(Fx \to Gx)$	$\exists$I, 6

1.2 $\forall x \neg Fx \vdash \neg\exists x\, Fx$

Strategy. Assume $\exists x\, Fx$ for reductio. By $\exists$E with a fresh constant a, we have Fa; the premise gives $\neg Fa$, yielding $\bot$.

1	$\forall x \neg Fx$	
2	$\mid\ \exists x\, Fx$	
3	$\mid\ \mid\ Fa$	
4	$\mid\ \mid\ \neg Fa$	$\forall$E, 1
5	$\mid\ \mid\ \bot$	$\neg$E, 4, 3
6	$\mid\ \bot$	$\exists$E, 2, 3–5
7	$\neg\exists x\, Fx$	$\neg$I, 2–6

1.3 $\neg\exists x\, Fx \vdash \forall x \neg Fx$

Strategy. For an arbitrary a, assume Fa; then $\exists x\, Fx$ by $\exists$I, contradicting the premise. Discharge to obtain $\neg Fa$, then generalise.

1	$\neg\exists x\, Fx$	
2	$\mid\ Fa$	
3	$\mid\ \exists x\, Fx$	$\exists$I, 2
4	$\mid\ \bot$	$\neg$E, 1, 3
5	$\neg Fa$	$\neg$I, 2–4
6	$\forall x\, \neg Fx$	$\forall$I, 5

1.4 $\exists x\, \neg Fx \vdash \neg\forall x\, Fx$

Strategy. Assume $\forall x\, Fx$ for reductio. Inside an $\exists$E subproof for a witness a with $\neg Fa$, instantiate the assumption to Fa and derive $\bot$.

1	$\exists x\, \neg Fx$	
2	$\mid\ \forall x\, Fx$	
3	$\mid\ \mid\ \neg Fa$	
4	$\mid\ \mid\ Fa$	$\forall$E, 2
5	$\mid\ \mid\ \bot$	$\neg$E, 3, 4
6	$\mid\ \bot$	$\exists$E, 1, 3–5
7	$\neg\forall x\, Fx$	$\neg$I, 2–6

1.5 $\neg\forall x\, Fx \vdash \exists x\, \neg Fx$

Strategy. This sequent requires classical reasoning. Assume $\neg\exists x\, \neg Fx$ for reductio. For arbitrary a, the inner assumption $\neg Fa$ immediately gives $\exists x\, \neg Fx$, contradicting the outer assumption. So $\neg\neg Fa$, whence Fa by $\neg\neg$E. Generalising yields $\forall x\, Fx$, contradicting the premise. A final application of $\neg\neg$E delivers the conclusion.

1	$\neg\forall x\, Fx$	
2	$\mid\ \neg\exists x\, \neg Fx$	
3	$\mid\ \mid\ \neg Fa$	
4	$\mid\ \mid\ \exists x\, \neg Fx$	$\exists$I, 3
5	$\mid\ \mid\ \bot$	$\neg$E, 2, 4
6	$\mid\ \neg\neg Fa$	$\neg$I, 3–5
7	$\mid\ Fa$	$\neg\neg$E, 6
8	$\mid\ \forall x\, Fx$	$\forall$I, 7
9	$\mid\ \bot$	$\neg$E, 1, 8
10	$\neg\neg\exists x\, \neg Fx$	$\neg$I, 2–9
11	$\exists x\, \neg Fx$	$\neg\neg$E, 10

1.6 $\forall x\,(Fx \to \forall y\,Fy) \vdash \forall x\,\forall y\,(Fx \to Fy)$

Note. The exercise sheet prints the premise as $\forall x(Fx \to \forall y\,Gy)$. As stated with two distinct predicate letters the sequent is not valid (take $F = \{1\}$, $G = \{1, 2\}$ in a two-element domain). The intended reading is almost certainly the one solved below, with F throughout.

Strategy. Instantiate the premise to a, obtaining $Fa \to \forall y\,Fy$. Assume Fa; derive $\forall y\,Fy$ by ⇒E and instantiate to b. Discharge twice and generalise over both variables.

1	$\forall x\,(Fx \to \forall y\,Fy)$	
2	$Fa \to \forall y\,Fy$	∀E, 1
3	$\quad\mid\ Fa$	
4	$\quad\mid\ \forall y\,Fy$	⇒E, 2, 3
5	$\quad\mid\ Fb$	∀E, 4
6	$Fa \to Fb$	⇒I, 3–5
7	$\forall y\,(Fa \to Fy)$	∀I, 6
8	$\forall x\,\forall y\,(Fx \to Fy)$	∀I, 7

The eigenvariable condition is satisfied at line 7 (b does not occur free in any undischarged assumption or in Fa) and at line 8 (a does not occur free in any undischarged assumption).

2 Identity

In the proofs below we use two primitive rules for identity:

- **=I** (reflexivity): derive $t = t$ for any term t, with no premises.
- **=E** (Leibniz): from $s = t$ and $\varphi(s)$, derive $\varphi(t)$ (substituting t for some occurrences of s in φ), or symmetrically.

2.1 $\vdash \forall x\,(x = x)$

1	$a = a$	=I
2	$\forall x\,(x = x)$	∀I, 1

2.2 $\vdash \forall x\,\forall y\,(x = y \to y = x)$

Strategy. Assume $a = b$. By =I we have $a = a$. Applying =E with $a = b$ to the formula $a = a$ (replacing the first occurrence of a by b) yields $b = a$.

1	$\quad\mid\ a = b$	
2	$\quad\mid\ a = a$	=I
3	$\quad\mid\ b = a$	=E, 1, 2
4	$a = b \to b = a$	⇒I, 1–3
5	$\forall y\,(a = y \to y = a)$	∀I, 4
6	$\forall x\,\forall y\,(x = y \to y = x)$	∀I, 5

2.3 $\vdash \forall x\, \forall y\, \forall z\, ((x = y \land y = z) \to x = z)$

Strategy. From $a = b$ and $b = c$, apply $=$E: in the formula $b = c$, replace b by a using $a = b$, obtaining $a = c$.

$$
\begin{array}{r|l|l}
1 & \quad \begin{array}{|l} a = b \land b = c \\ \hline \end{array} & \\
2 & \quad \begin{array}{|l} a = b \end{array} & \land\text{E, 1} \\
3 & \quad \begin{array}{|l} b = c \end{array} & \land\text{E, 1} \\
4 & \quad \begin{array}{|l} a = c \end{array} & =\text{E, 2, 3} \\
5 & (a = b \land b = c) \to a = c & \Rightarrow\text{I, 1–4} \\
6 & \forall z\, ((a = b \land b = z) \to a = z) & \forall\text{I, 5} \\
7 & \forall y\, \forall z\, ((a = y \land y = z) \to a = z) & \forall\text{I, 6} \\
8 & \forall x\, \forall y\, \forall z\, ((x = y \land y = z) \to x = z) & \forall\text{I, 7}
\end{array}
$$

3 Inconsistency

We prove the logical equivalence of the following three conditions on a set of formulas Γ:

(a) Γ is inconsistent, i.e. $\Gamma \vdash \bot$.
(b) For every sentence φ, $\Gamma \vdash \varphi$.
(c) For some sentence φ, $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.

(a) $\Rightarrow$ (b). Suppose $\Gamma \vdash \bot$. Let φ be any sentence. Since $\bot$ has been derived from Γ, one application of $\bot$E yields φ. Hence $\Gamma \vdash \varphi$.

$$
\begin{array}{r|ll}
1 & \bot & \text{from } \Gamma \\
2 & \varphi & \bot\text{E, 1}
\end{array}
$$

(b) $\Rightarrow$ (c). Immediate: (b) asserts that $\Gamma \vdash \varphi$ for *every* sentence φ. In particular, for a fixed sentence ψ we have both $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$.

(c) $\Rightarrow$ (a). Suppose $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ for some φ. Then both φ and $\neg\varphi$ are derivable from Γ, and one application of $\neg$E yields $\bot$. Hence $\Gamma \vdash \bot$, i.e. Γ is inconsistent.

$$
\begin{array}{r|ll}
1 & \varphi & \text{from } \Gamma \\
2 & \neg\varphi & \text{from } \Gamma \\
3 & \bot & \neg\text{E, 1, 2}
\end{array}
$$

4 Two useful semantic facts

4.1 Semantic monotonicity

If $\Gamma_i \subseteq \Gamma_j$ and $\Gamma_i \models \varphi$, then $\Gamma_j \models \varphi$.

Proof. Let $\mathcal{M}$ be an arbitrary model such that $\mathcal{M} \models \Gamma_j$, i.e. $\mathcal{M}$ satisfies every formula in Γ_j. Since $\Gamma_i \subseteq \Gamma_j$, every formula in Γ_i belongs to Γ_j, so $\mathcal{M} \models \Gamma_i$. By the hypothesis $\Gamma_i \models \varphi$, every model of Γ_i is a model of φ; hence $\mathcal{M} \models \varphi$. Because $\mathcal{M}$ was an arbitrary model of Γ_j, we conclude $\Gamma_j \models \varphi$. □

4.2 Semantic deduction theorem

If $\Gamma \cup \{\varphi\} \models \psi$, then $\Gamma \models \varphi \to \psi$.

Proof. Let $\mathcal{M}$ be an arbitrary model of Γ. We must show $\mathcal{M} \models \varphi \to \psi$, i.e. that if $\mathcal{M} \models \varphi$ then $\mathcal{M} \models \psi$.

Case 1: $\mathcal{M} \not\models \varphi$. Then $\varphi \to \psi$ is satisfied in $\mathcal{M}$ (a conditional with a false antecedent is true), so $\mathcal{M} \models \varphi \to \psi$.

Case 2: $\mathcal{M} \models \varphi$. Then $\mathcal{M}$ satisfies every formula in $\Gamma \cup \{\varphi\}$. By hypothesis, $\Gamma \cup \{\varphi\} \models \psi$, so $\mathcal{M} \models \psi$. Hence $\mathcal{M} \models \varphi \to \psi$.

In either case $\mathcal{M} \models \varphi \to \psi$, and since $\mathcal{M}$ was arbitrary, $\Gamma \models \varphi \to \psi$. □

5 Soundness

We prove that the introduction and elimination rules for $\bot$ and $\vee$ are sound, i.e. that each rule preserves semantic validity. In every case we suppose that the premises of the rule are semantically valid (with respect to a set of formulas Γ) and show that the conclusion is too.

5.1 $\bot$-Introduction ($\neg$-Elimination)

Rule. From $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, infer $\Gamma \vdash \bot$.

Soundness. Suppose $\Gamma \models \varphi$ and $\Gamma \models \neg\varphi$. Let $\mathcal{M}$ be any model of Γ. Then $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \neg\varphi$, i.e. $\mathcal{M} \not\models \varphi$ — a contradiction. Therefore no model satisfies Γ. It follows *vacuously* that every model of Γ satisfies $\bot$, i.e. $\Gamma \models \bot$.

5.2 $\bot$-Elimination (Ex Falso Quodlibet)

Rule. From $\Gamma \vdash \bot$, infer $\Gamma \vdash \varphi$ for any sentence φ.

Soundness. Suppose $\Gamma \models \bot$. Then no model satisfies Γ (since $\bot$ is false in every model). The statement "every model of Γ satisfies φ" is therefore *vacuously true* for any φ. Hence $\Gamma \models \varphi$.

5.3 $\vee$-Introduction

Rule. From $\Gamma \vdash \varphi$, infer $\Gamma \vdash \varphi \vee \psi$ (and symmetrically for the right disjunct).

Soundness. Suppose $\Gamma \models \varphi$. Let $\mathcal{M}$ be any model of Γ. Then $\mathcal{M} \models \varphi$. By the truth conditions for disjunction, $\mathcal{M} \models \varphi \vee \psi$. Since $\mathcal{M}$ was arbitrary, $\Gamma \models \varphi \vee \psi$. The argument for the right disjunct is symmetric.

5.4 $\vee$-Elimination (Proof by Cases)

Rule. From $\Gamma \vdash \varphi \vee \psi$ and subproofs $\Gamma, \varphi \vdash \chi$ and $\Gamma, \psi \vdash \chi$, infer $\Gamma \vdash \chi$.

Soundness. Suppose $\Gamma \models \varphi \vee \psi$, $\Gamma \cup \{\varphi\} \models \chi$, and $\Gamma \cup \{\psi\} \models \chi$. Let $\mathcal{M}$ be any model of Γ. Then $\mathcal{M} \models \varphi \vee \psi$, so $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \psi$.

- If $\mathcal{M} \models \varphi$, then $\mathcal{M}$ satisfies $\Gamma \cup \{\varphi\}$, so by hypothesis $\mathcal{M} \models \chi$.
- If $\mathcal{M} \models \psi$, then $\mathcal{M}$ satisfies $\Gamma \cup \{\psi\}$, so by hypothesis $\mathcal{M} \models \chi$.

In both cases $\mathcal{M} \models \chi$. Since $\mathcal{M}$ was arbitrary, $\Gamma \models \chi$.