# Natural Deduction for First-order Logic

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## 1 Quantification

Use the Natural Deduction rules to show that:

1. $\forall x \neg Fx \vdash \exists x(Fx \to Gx)$
2. $\forall x \neg Fx \vdash \neg \exists x Fx$
3. $\neg \exists x Fx \vdash \forall x \neg Fx$
4. $\exists x \neg Fx \vdash \neg \forall x Fx$
5. $\neg \forall x Fx \vdash \exists x \neg Fx$
6. $(\forall x)(Fx \to (\forall y)Gy) \vdash (\forall x)(\forall y)(Fx \to Fy)$

## 2 Identity

Prove the following theorems for FOL with identity:

1. $\vdash \forall x(x = x)$
2. $\vdash \forall x \forall y(x = y \to y = x)$
3. $\vdash \forall x \forall y \forall z((x = y \wedge y = z) \to x = z)$

## 3 Inconsistency

Prove that the following statements are logically equivalent:

- the set of formulas $\Gamma$ is inconsistent
- for any sentence $\phi$, $\Gamma \vdash \phi$
- for some sentence $\phi$, $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$

# 4 Two useful semantic facts

- **Semantic monotonicity**: Prove that: if $\Gamma_i \subseteq \Gamma_j$ and $\Gamma_i \models \phi$, then $\Gamma_j \models \phi$.
- Prove that: If $\Gamma \cup \{\phi\} \models \psi$, then $\Gamma \models \phi \to \psi$

# 5 Soundness

Prove that the introduction and elimination rules for $\bot$, as well as the introduction and elimination rules for $\vee$, are sound.