Proofs for first-order logic

Part 2: Natural Deduction

I Rules for quantifiers

∃-introduction

- **Existential generalization**

- t is a closed term; the rule states that from a true instance, an existential sentence can be inferred.

- So: from $Pa, \exists x Px$ can be deduced

- Illustration: let us show that: $Aa \to Bb \vdash \exists x \exists y (Ax \to By)$

$$\vdots$$

$$n \quad \varphi t$$

$$\vdots$$

$$m \quad \exists x\, \varphi x \qquad I\exists, 3$$

Illustration

1. $Aa \to Bb$ P
2. $\exists y(Aa \to By)$ I$\exists$, 1
3. $\exists x \exists y(Ax \to By)$ I$\exists$, 2

∀-elimination

- Universal instanciation
- Again, t is a closed term; the rule states that a universal sentence can be instantiated to any closed term.
- For instance: from $\forall x Px$, infer Pa.
- Illustration : $\forall x Px \vdash \exists x Px$

$$\vdots$$

$$m \qquad \forall x\, \varphi x$$

$$\vdots$$

$$n \qquad \varphi t \qquad E\forall, m$$

Illustration

1	$\forall x Px$	P
2	Pa	E$\forall$, 1
3	$\exists x Px$	I$\exists$, 2

Discussion: why can't $I(P)$ be empty if $\forall x Px$ is true?

- Let us prove now that : « Every human is moral. Socrates is human. Therefore Socrates is mortal » if valid.

- Formalization : $\forall x(Hx \to Mx), Ha \vdash Ma$

1	$\forall x(Hx \to Mx)$	P
2	Ha	P
3	$Ha \to Ma$	E$\forall$,1
4	Ma	MP,2,3

∀-introduction

- **Core idea: to prove a universal sentence, one has to prove every instance of it …**
- … but that's not feasible; so we prove an instance with an **arbitrary name**.
- From a proof-theoretic point of view, a name a is arbitrary iff: it does not occur in the conclusion $\forall x \phi x$, nor in the premises, nor in any assumption undischarged in the derivation ending with ϕa.
    - More concretely: check that you could replace the name you chose by any other name without altering the proof.
- Illustration : $\neg \exists x(Fx \wedge Gx) \vdash \forall x(Fx \rightarrow \neg Gx)$

$$\begin{array}{ll} & \vdots \\ m & \varphi a \\ & \vdots \\ n & \forall x \varphi x \end{array}$$

1	$\neg\exists x(Fx \wedge Gx)$	P
2	Fa	H, : [illegible] $\neg Ga$
3	Ga	~~H~~
4	$Fa \wedge Ga$	I$\wedge$, 2, 3
5	$\exists x(Fx \wedge Gx)$	I$\exists$, 4
6	$\perp$	I$\perp$, 1, 5
7	$\neg Ga$	I$\neg$, 3-6
8	$Fa \to \neg Ga$	I$\to$, 2-7
9	$\forall x(Fx \to \neg Gx)$	

The constant a is arbitrary in the proof!

I

∃-elimination

- Core idea: an existentially quantified sentence can be seen as a infinite disjunction (of all instances)

- So we use the ∨-elimination rule:

    - We assume that an arbitrary instance of $\exists x \phi x \psi$ is true, and from it we infer ψ;

    - We discharge the assumption and we assert ψ.

- a is arbitrary : it does not occur in the conclusion ψ, nor in the premises, nor in any assumption undischarged in the derivation ending with the assumption ϕa.

n $\exists x \varphi x$

m φa Ha

ψ

l ψ E∃ n,l

$a \rightarrow$ arbitrary

Illustration

- $\exists x(Fx \wedge Gx) \vdash \exists x Fx$

1	$\exists x(Fx \wedge Gx)$	P
2	$Fa \wedge Ga$	~~Ha~~
3	Fa	$E\wedge$, 2
4	$\exists x Fx$	$I\exists$, 3
5	$\exists x Fx$	$E\exists$, 2-4

Why the conditions are essential

- Let us consider the argument : « Someone is walking, Socrates is whistling, therefore someone is walking and whisling ».

- Formalization : $\exists xFx, Ga \nvdash \exists x(Fx \wedge Gx)$

INCORRECT !!

1	$\exists x Fx$	P
2	Ga	P
3	Fa	$\cancel{Ha}$
4	$Fa \wedge Ga$	I$\wedge$, 2, 3
5	$\exists x(Fx \wedge Ga)$	I$\exists$
6	$\exists x(Fx \wedge Gx)$	E$\exists$

There is a problem at line 3: the instantiation name a is not arbitrary! You can check that by using b instead: the proof is affected

- Now let us consider: « Someone is walking. Everyone who is walking is also whistling. Therefore Socrates is whistling ».

- Formalization : $\exists x Fx, \forall x(Fx \to Gx) \nvdash Ga$

INCORRECT !!

1	$\exists x Fx$	P
2	$\forall x(Fx \to Gx)$	P
3	$\vert$ $\underline{Fa}$	Hta
4	$\vert$ $Fa \to Ga$	$E\forall, 2$
5	$\vert$ Ga	MP, 3, 4
6	Ga	$\exists E, 1, 3\text{-}5$

There is a problem at line 6 : « a » is not arbitrary, it occurs in « Ga » !

II Rules for identity

=-Introduction

- For any close term t, we know that t=t ;
- So that's something we can introduce anywhere in a proof.

$\vdots$

$n \quad t = t \quad I=$

$\vdots$

=-elimination

- In order to eliminate an identity statement $t_1 = t_2$, we use Leibniz's law
    - That is: we can substitute t_2 for t_1 *salva veritate*
- *NB: this won't work for some English linguistic contexts.*
    - For instance : Maya believes that Eric Rohmer directed *Ma nuit chez Maud*; Eric Rohmer=Maurice Schérer; but it's not true that Maya believes that Maurice Schérer directed *Ma nuit chez Maud*

k	$t_1 = t_2$		$t_1 = t_2$
l	φt_1	l	φt_2
m	φt_2		φt_1 =E

Problem

- Prove the following theorems of FOL with identity:
  - $\vdash \forall x(x = x)$
  - $\vdash \forall x \forall y(x = y \rightarrow y = x)$
  - $\vdash \forall x \forall y \forall z((x = y \land y = z) \rightarrow x = z)$
- Hint: prove the theorems with arbitrary names, and then be careful to generalize step by step!

III Some proof-theoretic notions

Theorems, derivability, consistency

- A sentence ϕ is a **theorem** iff there is a derivation of ϕ from an empty set of premises in which all assumptions are discharged. Written: $\vdash \phi$; $\nvdash \phi$ means that ϕ is not a theorem.

- A sentence ϕ is **derivable** from a set of sentences Γ, written $\Gamma \vdash \phi$, iff there is a derivation with conclusion ϕ and in which every assumption is either discharged or en element of Γ. Written: $\Gamma \vdash \phi$; $\Gamma \nvdash \phi$ means that ϕ is not derivable from Γ.

- A set of sentences Γ is **inconsistent** iff $\Gamma \vdash \bot$; if $\Gamma \nvdash \bot$ we say that Γ is consistent.

Reflexivity and monotonicity

- **Reflexivity**: If $\phi \in \Gamma$ then $\Gamma \vdash \phi$. Let us consider a proof whose only line is the assumption ϕ. Since $\phi \in \Gamma$, it is a derivation of ϕ.

- **Syntactic** m**onotonicity**: if $\Gamma \subseteq \Delta$ and $\Gamma \vdash \phi$, then $\Delta \vdash \phi$. Any derivation of ϕ from Γ is also a derivation of ϕ from Δ.

- **Transitivity**: If $\Gamma \vdash \phi$ and $\Delta \cup \{\phi\} \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$

Proof

$\Delta, [\varphi]$

$\vdots$

ψ

$\varphi \to \psi$

$\vdots$

Γ

$\vdots$

φ

$\vdots$

ψ

All undischarged assumptions are in $\Delta \cup \Gamma$

Inconsistency

- The following are equivalent:
  - Γ is inconsistent
  - For any sentence ϕ, $\Gamma \vdash \phi$
  - For some ϕ, $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$
- Proof: exercise!

Derivability and inconsistency

- $\Gamma \cup \{\neg\phi\}$ is inconsistent iff $\Gamma \vdash \phi$

- $\Rightarrow$: Assume $\Gamma \cup \{\neg\phi\}$ is inconsistent, so: if we add the assumption $\neg\phi$ to Γ, we can derive $\bot$. In this context we can use the rule to introduce $\neg$: we discharge the assumption $\neg\phi$, and we write $\neg\neg\phi$. We then use the strong elimination of $\neg$ to get ϕ. Hence, $\Gamma \vdash \phi$.

- $\Leftarrow$: Assume $\Gamma \vdash \phi$. From $\Gamma \cup \{\neg\phi\}$: $\neg\phi$ can be derived (we use the repetition rule); ϕ can be derived (monotony) ; so $\bot$ can be derived.

Compactness

- If $\Gamma \vdash \phi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \phi$.
  - If $\Gamma \vdash \phi$, there is a derivation from Γ to the conclusion ϕ, with a set of undischarged assumptions that are all in Γ. But a proof is a finite sequence of lines. So the set of undischarged assumption is a finite set. Let us call it Γ_0. We see that there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \phi$.
- Note the contrapositive: If no finite subset $\Gamma_0 \subseteq \Gamma$ is such that $\Gamma_0 \vdash \phi$, then $\Gamma \nvdash \phi$. If we replace ϕ by $\bot$, we get :
- If no finite subset $\Gamma_0 \subseteq \Gamma$ is such that $\Gamma_0 \vdash \bot$, then $\Gamma \nvdash \bot$; or in words : **if no finite subset of Γ is inconsistent, Γ is consistent.**

IV Soundness theorem for Natural Deduction

Philosophical relevance

- $\Gamma \vdash \phi \Rightarrow \Gamma \models \phi$
- Formal computations preserve truth, which suggests that rational thought could be simulated by computations.
- Cf Jerry Fodor and the language of thought hypothesis: human thought as a semantic engine driven by syntax.

Preliminary lemma

- We'll write Γ_i the set of undischarged assumptions at line i of a proof, and ϕ_i the formula occurring at this line.
- **Lemma 1, Semantic monotonicity**: if $\Gamma_i \subseteq \Gamma_j$ and $\Gamma_i \vDash \phi$, then $\Gamma_j \vDash \phi$.
  - If $\Gamma_j \nvDash \phi$, there is a counter-model M such (i) every formula in Γ_j is true in M, but ϕ is false in M. But since $\Gamma_i \subseteq \Gamma_j$, this means that every formula in Γ_i is also true in M and ϕ is false in M, hence $\Gamma_i \nvDash \phi$, which contradicts our assumption. So it's false that $\Gamma_j \nvDash \phi$ and we can conclude that $\Gamma_i \vDash \phi$.
- **Lemma 2**: If $\Gamma \cup \{\phi\} \vDash \psi$, then $\Gamma \vDash \phi \to \psi$. Let us assume that $\Gamma \nvDash \phi \to \psi$. There is a counter-model M such that every formula in Γ is true in M, but $\phi \to \psi$ is false in M. Hence $M \vDash \phi$ but $M \nvDash \psi$. Since M satisfies ϕ in addition to every formula in Γ, it satisfies $\Gamma \cup \{\phi\}$. We have assumed that $\Gamma \cup \{\phi\} \vDash \psi$. So $M \vDash \psi$. Contradiction. So by reductio: it's false that $\Gamma \nvDash \phi \to \psi$, and therefore $\Gamma \vDash \phi \to \psi$

Proof strategy

- We want to prove that $\Gamma \vdash_{DN} \phi \Rightarrow \Gamma \vDash \phi$.
- We'll reason by induction:
  - We'll first show that for **proofs that are only one line long**;
  - Then we'll make an **inductive assumption**: for $k < n$, $\Gamma_k \vdash_{DN} \phi_k \Rightarrow \Gamma_k \vDash \phi_k$
  - And we'll show that on this assumption, $\Gamma_n \vdash_{DN} \phi_n \Rightarrow \Gamma_n \vDash \phi_n$

First inductive step

- In a one line proof, an undischarged assumption ϕ is also the conclusion of the proof. So $\Gamma_1 = \{\phi\}$. Do we have $\{\phi\} \vDash \phi$? Yes, since every model that satisfies ϕ (obviously) satisfies ϕ.

- So at this step we have: $\Gamma_1 \vdash_{DN} \phi \Rightarrow \Gamma_1 \vDash \phi$

Inductive assumption

- Now, let us assume that for $k < n$, $\Gamma_k \vdash_{DN} \phi_k \Rightarrow \Gamma_k \vDash \phi_k$
- We also assume that $\Gamma_n \vdash_{DN} \phi_n$. In order to show that $\Gamma_n \vDash \phi_n$, we have to build a proof for each rule that can be used to extend an (n-1) lines proof to a n lines proof, so actually: 15 proofs without =, or 18 proofs with =.

Repetition

- If we just repeat Γ_k to get the conclusion: $\Gamma_k \vDash \phi_k$ (inductive hypothesis), $\Gamma_k \subseteq \Gamma_n$, and so (monotonicity) $\Gamma_n \vDash \phi_n$

$$\begin{array}{ccc} & \vdots & \\ k & \phi_k & \Gamma_k \\ n & \phi_n & \Gamma_n \end{array}$$

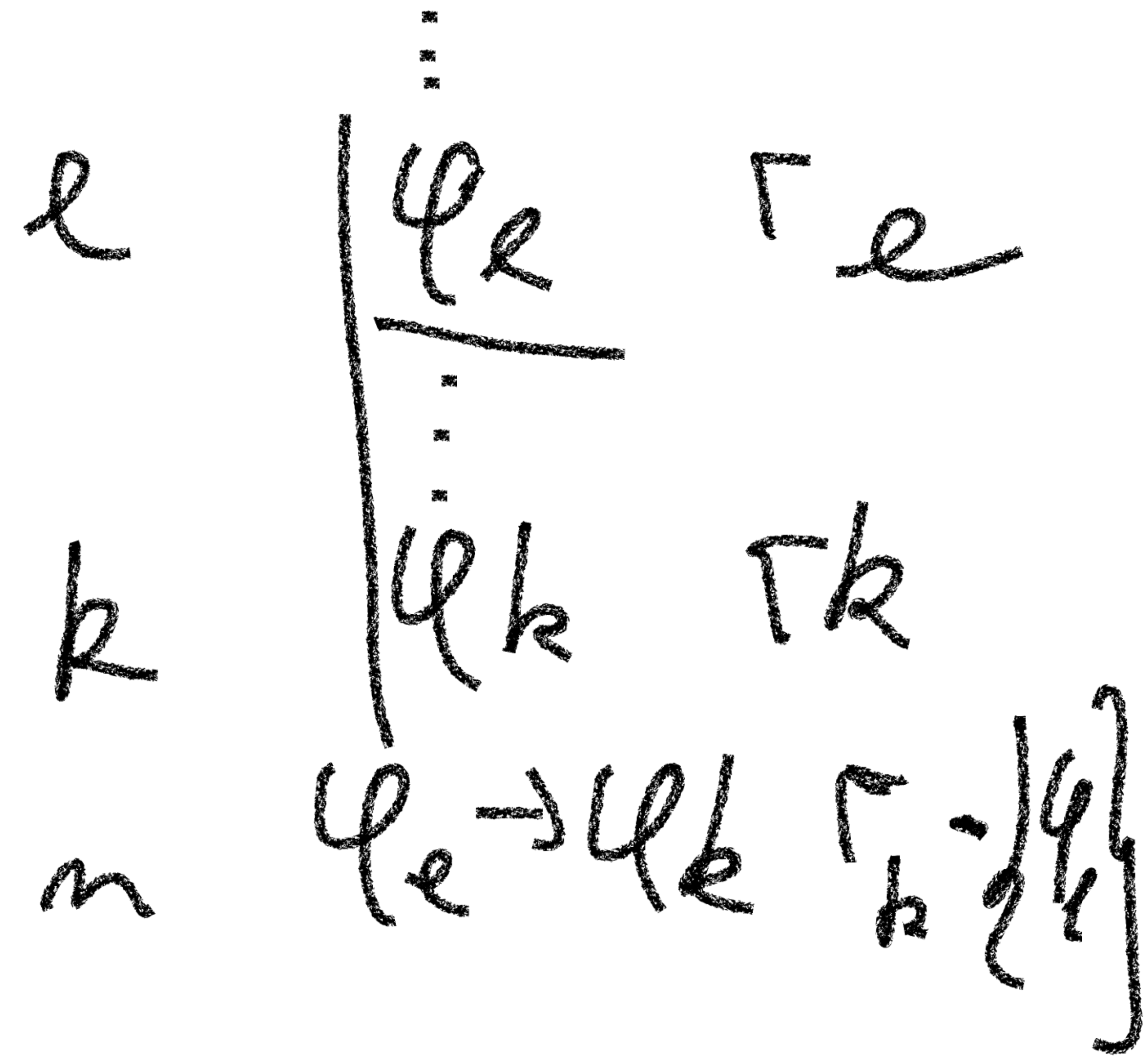
Conjunction introduction

- Induction hypothesis: $\Gamma_k \vdash \phi_k$ so $\Gamma_k \models \phi_k$, $\Gamma_l \vdash \phi_l$so $\Gamma_l \models \phi_l$. We also have : $\Gamma_k \subseteq \Gamma_l \subseteq \Gamma_n$. Hence:
- $\Gamma_n \models \phi_k$ and $\Gamma_l \models \phi_k$ (because of monotonicity). From the semantics of $\wedge$, it follows that:
- $\Gamma_n \models \phi_k \wedge \phi_l$

	$\vdots$	
k	φ_k	Γ_k
	$\vdots$	
l	φ_l	Γ_l
	$\vdots$	
n	$\underbrace{\varphi_k \wedge \varphi_l}_{\varphi_n}$	Γ_n

$\rightarrow$-introduction

- $\Gamma_l \subseteq \Gamma_k \subseteq \Gamma_n \cup \{\phi_n\}$
- $\Gamma_k \vdash \phi_k$, so because of the induction hypothesis, $\Gamma_k \vDash \phi_k$
- $\Gamma_k \subseteq \Gamma_n \cup \{\phi_l\}$, so because of monotony $\Gamma_n \cup \{\phi_l\} \vDash \phi_k$
- So because of Lemma 2, $\Gamma_n \vDash \phi_l \rightarrow \phi_k$



Universal generalization

- We want to show that $\Gamma_n \vDash \forall x \phi x$
- Now, $\Gamma_l \vdash \phi a$ and so because of the inductive hypothesis, $\Gamma_l \vDash \phi a$. Since $\Gamma_l \subseteq \Gamma_n$, $\Gamma_n \vDash \phi a$. So for all models M, if $M \vDash \Gamma_n$, then $M \vDash \phi a$
- Let us consider any model M such that $M \vDash \Gamma_n$.
- For any individual α in the domain, let us consider M', a model that only differs from M in that $I(a) = \alpha$. Since a does not occur in any element of Γ_n, $M' \vDash \Gamma_n$. So $M' \vDash \phi a$. So for any assignment σ, $M' \vDash_{\sigma[x \to \alpha]} \phi x$. And so $M' \vDash \forall x \phi x$. Since a does not occur in $\forall x \phi x$, $M \vDash \forall x \phi x$.
- Conclusion: for all M, if $M \vDash \Gamma_n$, $M \vDash \forall x \phi x$, and so $\Gamma_n \vDash \forall x \phi x$.

$\varphi k \quad \Gamma k$

$\vdots$

$\varphi a \quad \Gamma l$

$\forall x\, \varphi x \quad \Gamma_n$