Advanced Logic, 1
Solutions to Exercise Sheet 1

Notation and Definitions

Throughout these solutions, we use the notation and definitions from the course slides.

$\mathcal{L}$-Structures

- An **$\mathcal{L}$-structure** (or model) is a pair $\mathcal{M} = \langle D, I \rangle$ where:
  - D is a non-empty set called the *domain of discourse*,
  - I is an *interpretation function* which sends:
    * each constant symbol c to an object $I(c) \in D$,
    * each n-ary relation symbol R to an n-ary relation $I(R) \subseteq D^n$,
    * each n-ary function symbol f to a function $I(f) : D^n \to D$.
- A **variable assignment** (or simply: assignment) for a model $\mathcal{M} = \langle D, I \rangle$ is a function σ that assigns an object in D to each variable of the language.
- For an assignment σ, a variable x, and an object $o \in D$, we write $\sigma[x \to o]$ for the **variant of σ that assigns o to x**:

$$\sigma[x \to o](y) = \begin{cases} o & \text{if } y = x \\ \sigma(y) & \text{if } y \neq x \end{cases}$$

Interpretation of Terms

If I is an interpretation function and σ an assignment function, the **interpretation relative to the assignment** I_σ is defined as follows:

- If t is a variable: $I_\sigma(t) = \sigma(t)$
- If t is a constant symbol: $I_\sigma(t) = I(t)$
- If t is a complex term $f(t_1 \dots t_n)$: $I_\sigma(f(t_1 \dots t_n)) = I(f)(I_\sigma(t_1), \dots, I_\sigma(t_n))$

Satisfaction (Truth in a Model)

The **satisfaction relation** $\mathcal{M} \vDash_\sigma \varphi$ (read: "φ is true in $\mathcal{M}$ under assignment σ") is defined inductively:

Atomic formulae:

- $\mathcal{M} \vDash_\sigma t_i = t_j$ iff $I_\sigma(t_i) = I_\sigma(t_j)$
- $\mathcal{M} \vDash_\sigma R(t_1 \dots t_n)$ iff $\langle I_\sigma(t_1), \dots, I_\sigma(t_n) \rangle \in I(R)$

Propositional formulae:

- $\mathcal{M} \vDash_\sigma \neg\varphi$ iff $\mathcal{M} \nvDash_\sigma \varphi$
- $\mathcal{M} \vDash_\sigma (\varphi \wedge \psi)$ iff $\mathcal{M} \vDash_\sigma \varphi$ and $\mathcal{M} \vDash_\sigma \psi$
- $\mathcal{M} \vDash_\sigma (\varphi \vee \psi)$ iff $\mathcal{M} \vDash_\sigma \varphi$ or $\mathcal{M} \vDash_\sigma \psi$
- $\mathcal{M} \vDash_\sigma (\varphi \to \psi)$ iff $\mathcal{M} \nvDash_\sigma \varphi$ or $\mathcal{M} \vDash_\sigma \psi$
- $\mathcal{M} \vDash_\sigma (\varphi \leftrightarrow \psi)$ iff ($\mathcal{M} \vDash_\sigma \varphi$ iff $\mathcal{M} \vDash_\sigma \psi$)

Quantified formulae:

- $\mathcal{M} \vDash_\sigma \forall x\, \varphi$ iff for all variants $\sigma[x \to o]$ of σ: $\mathcal{M} \vDash_{\sigma[x\to o]} \varphi$
  (i.e., for all $o \in D$: $\mathcal{M} \vDash_{\sigma[x\to o]} \varphi$)
- $\mathcal{M} \vDash_\sigma \exists x\, \varphi$ iff for at least one variant $\sigma[x \to o]$ of σ: $\mathcal{M} \vDash_{\sigma[x\to o]} \varphi$
  (i.e., there exists $o \in D$ such that $\mathcal{M} \vDash_{\sigma[x\to o]} \varphi$)

Truth in a Model, Simpliciter

A formula φ is **true in** $\mathcal{M}$ (written $\mathcal{M} \vDash \varphi$) iff for any assignment σ: $\mathcal{M} \vDash_\sigma \varphi$.

Remark: For sentences (closed formulae), the truth value does not depend on the assignment.

Exercise 1

Setting: Let L be a language with signature $\{<\}$, and let $\varphi = x_1 < x_2$.

Part (a)

Consider $\mathcal{M} = (D, \bar{<})$ with $D = \{1, 3, 5\}$ and $\bar{<} = \{(1,3), (1,5), (3,5)\}$.

Here $\mathcal{M} = \langle D, I \rangle$ where $I(<) = \bar{<} = \{(1,3), (1,5), (3,5)\}$.

Solution. (i) Finding σ such that $\mathcal{M} \vDash_\sigma \varphi$:

Define the assignment σ by:

$$\sigma(x_1) = 1, \quad \sigma(x_2) = 3$$

(The values of σ on other variables are arbitrary.)

We verify:

$$\begin{aligned} \mathcal{M} \vDash_\sigma (x_1 < x_2) \quad &\text{iff} \quad \langle I_\sigma(x_1), I_\sigma(x_2) \rangle \in I(<) \\ &\text{iff} \quad \langle \sigma(x_1), \sigma(x_2) \rangle \in \bar{<} \\ &\text{iff} \quad \langle 1, 3 \rangle \in \bar{<} \end{aligned}$$

Since $\langle 1, 3 \rangle \in \{(1,3), (1,5), (3,5)\} = \bar{<}$, we conclude $\mathcal{M} \vDash_\sigma \varphi$.

(ii) Finding σ' such that $\mathcal{M} \nvDash_{\sigma'} \varphi$:

Define the assignment σ' by:

$$\sigma'(x_1) = 5, \quad \sigma'(x_2) = 1$$

We verify:

$$\begin{aligned} \mathcal{M} \vDash_{\sigma'} (x_1 < x_2) \quad &\text{iff} \quad \langle I_{\sigma'}(x_1), I_{\sigma'}(x_2) \rangle \in I(<) \\ &\text{iff} \quad \langle \sigma'(x_1), \sigma'(x_2) \rangle \in \bar{<} \\ &\text{iff} \quad \langle 5, 1 \rangle \in \bar{<} \end{aligned}$$

Since $\langle 5, 1 \rangle \notin \{(1,3), (1,5), (3,5)\} = \bar{<}$, we conclude $\mathcal{M} \nvDash_{\sigma'} \varphi$.

(iii) Do we have $\mathcal{M} \vDash \varphi$?

No, we do not have $\mathcal{M} \vDash \varphi$.

By definition, $\mathcal{M} \vDash \varphi$ iff for any assignment σ, $\mathcal{M} \vDash_\sigma \varphi$. However, we have exhibited in part (ii) an assignment σ' such that $\mathcal{M} \nvDash_{\sigma'} \varphi$.

Therefore, it is not the case that φ is true in $\mathcal{M}$. The formula $\varphi = x_1 < x_2$ contains free variables, and its truth value depends on the choice of assignment.

Part (b)

Let $\psi = \exists x(x < y)$. Find an assignment σ such that $\mathcal{M} \vDash_\sigma \psi$.

Solution. The formula $\psi = \exists x(x < y)$ has one free variable, namely y (the variable x is bound by the existential quantifier).

Define the assignment σ by:

$$\sigma(y) = 5$$

(The values of σ on other variables are arbitrary.)

By the semantic clause for existential quantification:

$$\mathcal{M} \vDash_\sigma \exists x(x < y) \quad \text{iff} \quad \text{for at least one variant } \sigma[x \to o] \text{ of } \sigma : \mathcal{M} \vDash_{\sigma[x \to o]} (x < y)$$

We need to find $o \in D$ such that $\mathcal{M} \models_{\sigma[x \to o]} (x < y)$.

Consider $o = 1$. We have:

$$\begin{aligned}
\mathcal{M} \models_{\sigma[x \to 1]} (x < y) \quad & \text{iff} \quad \langle I_{\sigma[x \to 1]}(x), I_{\sigma[x \to 1]}(y) \rangle \in I(<) \\
& \text{iff} \quad \langle \sigma[x \to 1](x), \sigma[x \to 1](y) \rangle \in \bar{<} \\
& \text{iff} \quad \langle 1, \sigma(y) \rangle \in \bar{<} \\
& \text{iff} \quad \langle 1, 5 \rangle \in \bar{<}
\end{aligned}$$

Since $\langle 1, 5 \rangle \in \bar{<}$, we have $\mathcal{M} \models_{\sigma[x \to 1]} (x < y)$.

Therefore, there exists a variant $\sigma[x \to o]$ of σ (namely with $o = 1$) such that $\mathcal{M} \models_{\sigma[x \to o]} (x < y)$, which means $\mathcal{M} \models_\sigma \psi$.

Remark: The assignment σ with $\sigma(y) = 3$ would also work (witnessed by $o = 1$). However, $\sigma(y) = 1$ would *not* work, since there is no $o \in D$ with $\langle o, 1 \rangle \in \bar{<}$.

Exercise 2

Setting: Consider a language $\mathcal{L}$ with three constant symbols a, b, c, a unary predicate letter A, and a binary predicate letter R.

Part (a): Signature of the Language

Solution. The signature of $\mathcal{L}$ specifies the non-logical vocabulary:

$$\text{Signature} = \{a, b, c, A, R\}$$

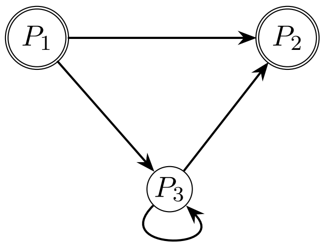
where:

- a, b, c are constant symbols,
- A is a unary (1-ary) relation symbol,
- R is a binary (2-ary) relation symbol.

There are no function symbols in this signature.

Part (b): Description of the Interpretation

Solution. Based on the diagram, the $\mathcal{L}$-structure $\mathcal{M} = \langle D, I \rangle$ is defined as follows:



Domain:

$$D = \{P_1, P_2, P_3\}$$

Interpretation function I:

Constants:

$$I(a) = P_1$$
$$I(b) = P_2$$
$$I(c) = P_3$$

Unary relation A (the circled points):

$$I(A) = \{P_1, P_2\}$$

Binary relation R (the arrow relation):

$$I(R) = \{(P_1, P_2), (P_1, P_3), (P_3, P_2), (P_3, P_3)\}$$

Thus, the complete structure is:

$$\mathcal{M} = \langle \{P_1, P_2, P_3\}, I \rangle$$

with I as specified above.

Part (c): Truth Values of Formulae in $\mathcal{M}$

Since all formulas in this part are *sentences* (closed formulas with no free variables), their truth values are independent of the choice of assignment. We fix an arbitrary assignment σ and evaluate each formula.

1. $\exists x \exists y \exists z(Rxy \wedge Ay \wedge \neg Az)$

Solution. Truth value: TRUE
By the semantic clause for $\exists$:

$$\begin{aligned} &\mathcal{M} \vDash_\sigma \exists x \exists y \exists z(Rxy \wedge Ay \wedge \neg Az) \\ &\quad \text{iff} \quad \text{for at least one variant } \sigma[x \to o_1] \text{ of } \sigma: \\ &\qquad \mathcal{M} \vDash_{\sigma[x \to o_1]} \exists y \exists z(Rxy \wedge Ay \wedge \neg Az) \end{aligned}$$

Unpacking all three existential quantifiers, this holds iff there exist $o_1, o_2, o_3 \in D$ such that:

$$\mathcal{M} \vDash_{\sigma[x \to o_1][y \to o_2][z \to o_3]} (Rxy \wedge Ay \wedge \neg Az)$$

Consider $o_1 = P_1$, $o_2 = P_2$, $o_3 = P_3$.
Let $\sigma' = \sigma[x \to P_1][y \to P_2][z \to P_3]$.
We verify each conjunct:

- $\mathcal{M} \vDash_{\sigma'} Rxy$ iff $\langle I_{\sigma'}(x), I_{\sigma'}(y) \rangle \in I(R)$ iff $\langle \sigma'(x), \sigma'(y) \rangle \in I(R)$ iff $\langle P_1, P_2 \rangle \in I(R)$.
  Since $(P_1, P_2) \in I(R)$: **TRUE**.
- $\mathcal{M} \vDash_{\sigma'} Ay$ iff $I_{\sigma'}(y) \in I(A)$ iff $\sigma'(y) \in I(A)$ iff $P_2 \in I(A)$.
  Since $P_2 \in \{P_1, P_2\} = I(A)$: **TRUE**.
- $\mathcal{M} \vDash_{\sigma'} \neg Az$ iff $\mathcal{M} \nvDash_{\sigma'} Az$ iff $\sigma'(z) \notin I(A)$ iff $P_3 \notin I(A)$.
  Since $P_3 \notin \{P_1, P_2\} = I(A)$: **TRUE**.

Since all conjuncts are satisfied under σ', and σ' is a variant of σ, the existential statement is satisfied.

2. $\forall x\, Rxx$

Solution. Truth value: FALSE
By the semantic clause for $\forall$:

$$\mathcal{M} \vDash_\sigma \forall x\, Rxx \quad \text{iff} \quad \text{for all variants } \sigma[x \to o] \text{ of } \sigma : \mathcal{M} \vDash_{\sigma[x \to o]} Rxx$$

We check each element of D:

For $o = P_1$:

$$\begin{aligned} \mathcal{M} \vDash_{\sigma[x \to P_1]} Rxx \quad &\text{iff} \quad \langle I_{\sigma[x \to P_1]}(x), I_{\sigma[x \to P_1]}(x) \rangle \in I(R) \\ &\text{iff} \quad \langle P_1, P_1 \rangle \in I(R) \end{aligned}$$

Since $(P_1, P_1) \notin I(R) = \{(P_1, P_2), (P_1, P_3), (P_3, P_2), (P_3, P_3)\}$: **FALSE**.

Since we found a variant $\sigma[x \to P_1]$ such that $\mathcal{M} \nvDash_{\sigma[x \to P_1]} Rxx$, the universal statement fails.

3. $\forall x(Rxx \leftrightarrow \neg Ax)$

Solution. Truth value: TRUE

We must check whether for all variants $\sigma[x \to o]$ of σ:

$$\mathcal{M} \vDash_{\sigma[x\to o]} (Rxx \leftrightarrow \neg Ax)$$

For the biconditional to hold, both sides must have the same truth value.

For $o = P_1$**:** Let $\sigma_1 = \sigma[x \to P_1]$.

- $\mathcal{M} \vDash_{\sigma_1} Rxx$ iff $\langle P_1, P_1 \rangle \in I(R)$. **FALSE**.
- $\mathcal{M} \vDash_{\sigma_1} \neg Ax$ iff $\mathcal{M} \nvDash_{\sigma_1} Ax$ iff $P_1 \notin I(A) = \{P_1, P_2\}$. **FALSE**.

The biconditional FALSE $\leftrightarrow$ FALSE evaluates to **TRUE**.

For $o = P_2$**:** Let $\sigma_2 = \sigma[x \to P_2]$.

- $\mathcal{M} \vDash_{\sigma_2} Rxx$ iff $\langle P_2, P_2 \rangle \in I(R)$. **FALSE**.
- $\mathcal{M} \vDash_{\sigma_2} \neg Ax$ iff $P_2 \notin I(A) = \{P_1, P_2\}$. **FALSE**.

The biconditional FALSE $\leftrightarrow$ FALSE evaluates to **TRUE**.

For $o = P_3$**:** Let $\sigma_3 = \sigma[x \to P_3]$.

- $\mathcal{M} \vDash_{\sigma_3} Rxx$ iff $\langle P_3, P_3 \rangle \in I(R)$. **TRUE** (since $(P_3, P_3) \in I(R)$).
- $\mathcal{M} \vDash_{\sigma_3} \neg Ax$ iff $P_3 \notin I(A) = \{P_1, P_2\}$. **TRUE**.

The biconditional TRUE $\leftrightarrow$ TRUE evaluates to **TRUE**.

Since the biconditional holds for all $o \in D$, the universal statement is true.

4. $\exists x \exists y(Rxy \wedge \neg Ax \wedge \neg Ay)$

Solution. Truth value: TRUE

We need to find $o_1, o_2 \in D$ such that:

$$\mathcal{M} \vDash_{\sigma[x\to o_1][y\to o_2]} (Rxy \wedge \neg Ax \wedge \neg Ay)$$

Since $I(A) = \{P_1, P_2\}$, the only element not in A is P_3.
Consider $o_1 = P_3$, $o_2 = P_3$. Let $\sigma' = \sigma[x \to P_3][y \to P_3]$.

- $\mathcal{M} \vDash_{\sigma'} Rxy$ iff $\langle \sigma'(x), \sigma'(y) \rangle \in I(R)$ iff $\langle P_3, P_3 \rangle \in I(R)$. **TRUE**.
- $\mathcal{M} \vDash_{\sigma'} \neg Ax$ iff $\sigma'(x) \notin I(A)$ iff $P_3 \notin I(A) = \{P_1, P_2\}$. **TRUE**.
- $\mathcal{M} \vDash_{\sigma'} \neg Ay$ iff $\sigma'(y) \notin I(A)$ iff $P_3 \notin I(A) = \{P_1, P_2\}$. **TRUE**.

All conjuncts are satisfied under σ', so the existential statement holds.

5. $\forall x(Rxx \to \exists y(Rxy \wedge Ay))$

Solution. Truth value: TRUE

We must verify that for all variants $\sigma[x \to o]$ of σ:

$$\mathcal{M} \vDash_{\sigma[x\to o]} (Rxx \to \exists y(Rxy \wedge Ay))$$

By the semantic clause for $\to$, this holds iff $\mathcal{M} \nvDash_{\sigma[x\to o]} Rxx$ or $\mathcal{M} \vDash_{\sigma[x\to o]} \exists y(Rxy \wedge Ay)$.

For $o = P_1$: Let $\sigma_1 = \sigma[x \to P_1]$.
$\mathcal{M} \vDash_{\sigma_1} Rxx$ iff $\langle P_1, P_1 \rangle \in I(R)$. **FALSE**.
Since the antecedent is false, the implication is *vacuously true*.

For $o = P_2$: Let $\sigma_2 = \sigma[x \to P_2]$.
$\mathcal{M} \vDash_{\sigma_2} Rxx$ iff $\langle P_2, P_2 \rangle \in I(R)$. **FALSE**.
Again, the implication is vacuously true.

For $o = P_3$: Let $\sigma_3 = \sigma[x \to P_3]$.
$\mathcal{M} \vDash_{\sigma_3} Rxx$ iff $\langle P_3, P_3 \rangle \in I(R)$. **TRUE**.
We must now verify the consequent: $\mathcal{M} \vDash_{\sigma_3} \exists y(Rxy \wedge Ay)$.
This holds iff for at least one variant $\sigma_3[y \to o']$: $\mathcal{M} \vDash_{\sigma_3[y \to o']} (Rxy \wedge Ay)$.
Consider $o' = P_2$. Let $\sigma_3' = \sigma_3[y \to P_2] = \sigma[x \to P_3][y \to P_2]$.

- $\mathcal{M} \vDash_{\sigma_3'} Rxy$ iff $\langle P_3, P_2 \rangle \in I(R)$. **TRUE**.
- $\mathcal{M} \vDash_{\sigma_3'} Ay$ iff $P_2 \in I(A) = \{P_1, P_2\}$. **TRUE**.

The consequent is satisfied, so the implication holds for $o = P_3$.

Since the implication holds for all $o \in D$, the universal statement is true.

6. $\forall x(Ax \to \exists y\, Rxy)$

Solution. Truth value: FALSE
We must verify that for all variants $\sigma[x \to o]$ of σ:

$$\mathcal{M} \vDash_{\sigma[x \to o]} (Ax \to \exists y\, Rxy)$$

For $o = P_1$: Let $\sigma_1 = \sigma[x \to P_1]$.
$\mathcal{M} \vDash_{\sigma_1} Ax$ iff $P_1 \in I(A) = \{P_1, P_2\}$. **TRUE**.
We must verify: $\mathcal{M} \vDash_{\sigma_1} \exists y\, Rxy$, i.e., for at least one variant $\sigma_1[y \to o']$: $\mathcal{M} \vDash_{\sigma_1[y \to o']} Rxy$.
Consider $o' = P_2$. Then:

$$\mathcal{M} \vDash_{\sigma_1[y \to P_2]} Rxy \quad \text{iff} \quad \langle P_1, P_2 \rangle \in I(R) \quad \textbf{TRUE.}$$

The consequent is satisfied, so the implication holds for $o = P_1$.

For $o = P_2$: Let $\sigma_2 = \sigma[x \to P_2]$.
$\mathcal{M} \vDash_{\sigma_2} Ax$ iff $P_2 \in I(A) = \{P_1, P_2\}$. **TRUE**.
We must verify: $\mathcal{M} \vDash_{\sigma_2} \exists y\, Rxy$, i.e., there exists $o' \in D$ such that $\langle P_2, o' \rangle \in I(R)$.
We check all possible values of $o' \in D$:

- $o' = P_1$: $\langle P_2, P_1 \rangle \in I(R)$? **FALSE**.
- $o' = P_2$: $\langle P_2, P_2 \rangle \in I(R)$? **FALSE**.
- $o' = P_3$: $\langle P_2, P_3 \rangle \in I(R)$? **FALSE**.

There is no $o' \in D$ such that $\langle P_2, o' \rangle \in I(R)$. Therefore, the consequent $\exists y\, Rxy$ is **FALSE**.
Since the antecedent is true and the consequent is false, the implication fails for $o = P_2$.

Since the implication fails for the variant with $o = P_2$, the universal statement is false.

7. $\exists x \exists y(Rxy \wedge \neg Ryx \wedge \exists z(Rxz \wedge Rzy))$

Solution. Truth value: TRUE

This formula asserts: there exist x, y such that R holds from x to y but not from y to x, and there is an intermediate element z forming a path $x \to z \to y$.

We need $o_1, o_2 \in D$ such that:

$$\mathcal{M} \vDash_{\sigma[x \to o_1][y \to o_2]} (Rxy \wedge \neg Ryx \wedge \exists z(Rxz \wedge Rzy))$$

Consider $o_1 = P_1$, $o_2 = P_2$. Let $\sigma' = \sigma[x \to P_1][y \to P_2]$.

- $\mathcal{M} \vDash_{\sigma'} Rxy$ iff $\langle P_1, P_2 \rangle \in I(R)$. **TRUE**.
- $\mathcal{M} \vDash_{\sigma'} \neg Ryx$ iff $\langle P_2, P_1 \rangle \notin I(R)$.
  Since $I(R) = \{(P_1, P_2), (P_1, P_3), (P_3, P_2), (P_3, P_3)\}$ and $(P_2, P_1) \notin I(R)$: **TRUE**.

Now we verify: $\mathcal{M} \vDash_{\sigma'} \exists z(Rxz \wedge Rzy)$.
This requires finding $o_3 \in D$ such that $\mathcal{M} \vDash_{\sigma'[z \to o_3]} (Rxz \wedge Rzy)$.
Consider $o_3 = P_3$. Let $\sigma'' = \sigma'[z \to P_3] = \sigma[x \to P_1][y \to P_2][z \to P_3]$.

- $\mathcal{M} \vDash_{\sigma''} Rxz$ iff $\langle \sigma''(x), \sigma''(z) \rangle \in I(R)$ iff $\langle P_1, P_3 \rangle \in I(R)$. **TRUE**.
- $\mathcal{M} \vDash_{\sigma''} Rzy$ iff $\langle \sigma''(z), \sigma''(y) \rangle \in I(R)$ iff $\langle P_3, P_2 \rangle \in I(R)$. **TRUE**.

All conditions are satisfied, so the existential statement is true.

Summary Table for Part (c)

#	**Formula**	**Truth Value**
1	$\exists x \exists y \exists z(Rxy \wedge Ay \wedge \neg Az)$	TRUE
2	$\forall x\, Rxx$	FALSE
3	$\forall x(Rxx \leftrightarrow \neg Ax)$	TRUE
4	$\exists x \exists y(Rxy \wedge \neg Ax \wedge \neg Ay)$	TRUE
5	$\forall x(Rxx \to \exists y(Rxy \wedge Ay))$	TRUE
6	$\forall x(Ax \to \exists y\, Rxy)$	FALSE
7	$\exists x \exists y(Rxy \wedge \neg Ryx \wedge \exists z(Rxz \wedge Rzy))$	TRUE